DECISION THEORY APPLIED TO THE SIMULATED DATA ACQUISITION AND MANAGEMENT OF A SALMON FISHERY[1]

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ABSTRACT

A salmon fishery management model utilizing statistical decision theory has been constructed. The model provides for the successive acquisition of data that can be used to formulate and maintain an optimum management strategy. The Bayes risk is defined as the expected economic loss resulting from a set of fishery management decisions and the criterion of optimality is taken to be the strategy that minimizes the Bayes risk. Specific functional forms are assumed where necessary in order to obtain a closed form expression for the Bayes risk. The Bayes risk, in units of numbers of fish, can then be computed for any particular sequence of fishery management decisions.

This paper represents a continuation of an earlier effort (Lord 1973) in which statistical decision theory was applied to the data acquisition and management of a salmon fishery. The crucial feature was not that the species considered was salmon but that the assumed fishery was both dynamic and subject to errors in the population estimation. The population is assumed to be subject to continuing assessment, however, so that as the season progresses it is possible to make repeatedly more refined estimates of the true state of nature. The management strategy may thus be modified successively to reflect the additional data as they become available.

The development was quite abstract and presented only the basic theory in a relatively general way. The present paper represents an intermediate situation in which the theory is applied to a specific model constructed to represent such a fishery. The principal features of this model are: 1) a Ricker spawner-return relationship, 2) simulated sampling for population estimation purposes, and 3) an economic loss function based on maximum substained yield (MSY).

A limitation of the present model is that it is constructed in such a manner that a closed analytic form is obtained without recourse to Monte Carlo or other approximate methods of analysis. In other words, the Bayes risk may be computed exactly upon the specification of well defined sets of parameters. The imposition of such analytical requirements constrains the choice of functions to those that are mathematically tractable. Anticipating the final results, Equations (18) and (20), I feel that about the maximum degree of generality has been retained consistent with analytical tractability. It is likely that models possessing a greater degree of fidelity to the actual fishery situations will require the use of Monte Carlo methods as Mathews (1966) used in his simulation of the cannery portion of the Bristol Bay fishery.

ANALYSIS

The notation used in Lord (1973), with only minor changes, will be retained here. In this section I will discuss the Bayes risk for a particular fisheries model based on the Ricker spawner-return relation. The criterion of optimality will be taken as MSY. Economic losses will accrue as the actual management strategies depart from the optimum. Generally these losses will be reflected in either a decreased present catch or in diminished future returns due to prior overfishing.

A loss function proportional to the difference between the optimum catch and the actual catch, on an MSY basis, will be assumed. This is a simple and intuitively reasonable concept but, nonetheless, a unique formulation of the loss function from this criterion is no simple task. The difficulty arises from the use of a spawner-return relation which reflects the biological fact that the present state of the system is necessarily the result of past actions and, similarly, that future conditions will depend on present actions. In the case of sockeye salmon,

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an added complication is provided by the fact that the run in any year represents the progeny of several spawning groups.

The Ricker spawner-return relation, for a single spawning group, is given by

$$R_n = aE_{n-k}e^{-bE_{n-k}} \tag{1}$$

where R_n is the return in year n resulting from an escapement E_{n-k} k years prior, e is the base of natural logarithms, and a and b are parameters assumed unique for any river system or spawning group. We may generalize Equation (1) to the case of multiple spawning groups to give

$$R_n = \sum_{k=1}^{K} a_k E_{n-k} e^{-bE_{n-k}} \tag{2}$$

where the relevant spawning occurs over the years $(n - 1)$ through $(n - K)$. In Equation (2) the coefficients $\{a_k\}$ now reflect not only the magnitude of the run, as in Equation (1), but the proportion of the run arising from each spawning group. Specifically we can write $a_k = ap_k$ where a is the parameter in Equation (1) and p_k is proportion of the run in year n arising from spawners in year $(n - k)$. We have the condition $\sum_{k=1}^{K} p_k = 1$ from which it follows that $\sum_{k=1}^{K} a_k = a$.

The return as given by Equation (1) or Equation (2) is a deterministic function of the parameters. In actual practice, however, the return, from the biologist's point of view, is a random variable in which case some additive or multiplicative error term must be appended to Equation (1). Thus, at an appropriate point in the analysis, the return will be assumed to be a random variable whose expected value is given by Equation (1).

Let X_n be the catch in year n. Then

$$X_n = a \sum_{k=1}^{K} p_k E_{n-k} e^{-bE_{n-k}} - E_n .$$

Let X_{tot} be the total catch over some fixed but otherwise arbitrary number, say n^*, of fishing seasons. Then

$$X_{tot} = \sum_{n=1}^{n^*} \left[\sum_{k=1}^{K} a_k E_{n-k} e^{-bE_{n-k}} - E_n \right]. \tag{3}$$

If we attempt to maximize X_{tot} with respect to the yearly escapements $E_1, E_2, \ldots E_{n^*}$, it turns out that as $n^* \to \infty$: a) a steady state solution exists and b) the optimum steady state escapement, E_0, is that which maximizes the function $(aEe^{-bE} - E)$.

Let L_n denote the economic loss in any year n and define L_n as the difference between the optimum catch, X_{opt}, and the actual catch X_{act}, i.e., $L_n = X_{opt} - X_{act}$. From Equation (3) we obtain

$$L_n = (aE_0e^{-bE_0} - E_0) - \left(a \sum_{k=1}^{K} p_k E_{n-k} e^{-bE_{n-k}} - E_n\right). \tag{4}$$

E_0 is fixed and all of the escapements $\{E_{n-k}\}(k = 1 \ldots K)$ have already occurred thus leaving only E_n at our disposal. L_n is clearly minimized by setting $E_n = 0$ but since this would eliminate a portion of the run in future years the subsequent loss would be high indeed. Consider now the combined loss for two successive years n and $(n + 1)$. Proceeding along the same lines that led to Equation (4) we obtain

$$L_n + L_{n+1} = 2(aE_0e^{-bE_0} - E_0) - \left(a \sum_{k=1}^{K} p_k E_{n-k} e^{-bE_{n-k}} - E_n\right) - \left(a \sum_{k=1}^{K} p_k E_{n+1-k} e^{-bE_{n+1-k}} - E_{n+1}\right).$$

If this is treated as a function of the single variable E_n, an optimum value can be obtained. However, this loss also depends on E_{n+1} which has not yet occurred. Let us extend this process through year $(n + K)$, which is a convenient stopping point since it represents the completion of a cycle starting at year n. The total loss over this period is given by

$$L_{n,n+K} = (K + 1)(aE_0e^{-bE_0} - E_0) - \sum_{j=n}^{n+K} \left[a \sum_{k=1}^{K} p_k E_{j-k} e^{-bE_{j-k}} - E_j \right]. \tag{5}$$

The loss given by Equation (5) depends not only on past and present escapements but on the future values $E_{n+1}, E_{n+2}, \ldots E_{n+K}$ as well. Thus, when formulating a policy for any particular year one must take into account future policies also. From a mathematical point of view what we have emerging here is another dynamic program, i.e., the optimum year-to-year allocation as well as the within-year allocation is in the form of a dynamic program. This is too great an anlytical burden to bear. However, we can invoke the "Principle of Optimality" (Bellman 1957:83) to specify that $E_{n+j} = E_0$ for all $j \geq 1$, i.e., all future escapements are assumed to be the optimum MSY escapement. This is a reasonable assumption since the principle of optimality states that an optimal policy is one which, given the present state of the system, establishes and maintains an optimal policy for all future time periods. Since E_0 represents such an optimum steady state escapement it follows that $E_{n+j} = E_0$ for future optimality. In this case Equation (5) takes the form

$$L_{n,n+K} = L(E_n) = (K+1)(aE_0e^{-bE_0} - E_0) - (aE_ne^{-bE_n} - E_n) + \text{(terms depending on } E_0 \text{ and past escapements only).} \quad (6)$$

From Equation (6) it appears that the optimization will be over a total of $(K + 1)$ seasons. This is not actually the case since, as noted above, the constraint $E_{n+j} = E_0$ has been imposed and the analytical procedures used in year n will be applicable in year $(n + 1)$, etc. Note also the intuitively reasonable result that the loss given by Equation (6) is minimized by setting $E_n = E_0$, the optimum MSY escapement.

The analysis thus far has assumed that all quantities are deterministic. Random variables will now be introduced to simulate the situation actually existing in salmon fishery assessment and management. Let N_n denote the run size resulting from the (known) escapements $\{E_{n-j}\}(j = 1, \ldots K)$ and let N_n be a random variable which, for definiteness, will be assumed to have the two-parameter gamma density

$$f_1(N_n \mid y_0) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} N_n^{\alpha_0 - 1} e^{-\beta_0 N_n} \quad (7)$$

where Γ denotes a gamma function. The parameters (α_0, β_0) are subscripted to denote that they are applicable prior to the start of the run and f is subscripted by one to denote that it is applicable to the first fishing period. The quantity y_0 is a symbolic conditioning variable denoting the preseason information that is available for the specification of (α_0, β_0). Anticipating the dynamic nature of the fishery and its management the probability density of N_n will be conditioned successively to reflect the data obtained after the start of the run.

We assume now that the expected value, shown as $E[N_n]$ is that given by the Ricker relation, i.e.,

$$R_n = E[N_n] = a \sum_{k=1}^{K} p_k E_{n-k} e^{-bE_{n-k}}. \quad (8)$$

The variance of N_n may be estimated from historical data, e.g., smolt outmigrations, high seas catches, etc. Knowledge of the mean and variance is sufficient to determine the parameters (α_0, β_0).

At this point it might be well to justify, or at least explain, the assumption of a gamma density for N_n. Clearly, one cannot obtain Equation (7) on the basis of biological arguments. On the other hand, a gamma density does not do particular violence to one's intuition concerning the distribution of population sizes. In particular, Equation (7) confines N_n to positive values with scale and location specified by (α_0, β_0). In salmon population estimation, it is rare that parameters beyond mean and variance are available from whatever source. It is in this spirit that Equation (7) is introduced. Further, the gamma distribution, not coincidentally, has the added virtue that it is an analytically convenient function. Similar arguments will be used to justify some of the functions to be introduced subsequently.

For the remainder of the analysis, only events in year n will be considered so that the subscript may be omitted from N_n. The fishing season is assumed to consist of m nonoverlapping time periods during each of which a management decision, δ, must be made. Let $\{\delta_i\}$ $(i = 1, \ldots m)$ be an arbitrary sequence of decisions where each of the δ_i is a member of some finite set of possible management decisions.[3] Assume now that during the ith period a fraction ρ_i of the total run enters the fishery. The set $\{\rho_i\}(i = 1, \ldots m)$, which is assumed to be known, may be obtained from such sources as the almanac prepared by Royce (1965). The $\{\rho_i\}$ must obviously satisfy the condition

$$\sum_{i=1}^{m} \rho_i = 1.$$

Corresponding to any actual realization of the run, N, there exists some unique set of optimum catch-escapement allocations $\{\eta_i\}$ $(i = 1, \ldots m)$. Rothschild and Balsiger (1971) used linear pro-

[3]A typical set of management decisions consists of such actions as opening or closing the fishery, the imposition of gear limitations, waiting periods, etc.

gramming to determine an optimum set of such allocations. Such fine-scale is not practical here so that the individual η_i are irrelevant here except that they must satisfy the condition

$$N \sum_{i=1}^{m} \rho_i \eta_i = E_0.$$

Let$\{\hat{\eta}_i\}$ $(i = 1, \ldots m)$ be the actual allocations where each $\hat{\eta}_i$ will be assumed to be a random function of the management decision δ_i taken during the ith period. It will be assumed that the $\{\hat{\eta}_i\}$ $(i = 1, \ldots m)$ have independent beta distributions where the beta parameters (ν_i, μ_i) are uniquely determined by the management decision δ_i that is taken during period i. Thus we have

$$g(\hat{\eta}_1, \hat{\eta}_2 \ldots \hat{\eta}_m \mid \delta_1, \delta_2, \ldots \delta_m) = \prod_{i=1}^{m} g_i(\hat{\eta}_i \mid \delta_i) \quad (9a)$$

where

$$g_i(\hat{\eta}_i \mid \delta_i) = \frac{\Gamma(\nu_i + \mu_i)}{\Gamma(\nu_i)\,\Gamma(\mu_i)} \hat{\eta}_i^{\nu_i - 1} (1 - \hat{\eta}_i)^{\mu_i - 1}. \quad (9b)$$

This is a reasonable probability density to assume since it confines $\hat{\eta}_i$ to the interval (0,1) and the parameter choice permits, within appropriate limits, the specification of the mean and variance[4] of $\hat{\eta}_i$.

Return now to the central feature of the analysis which is to take into account the dynamics of the fishery. Equation (7) is the probability density of N appropriate for the first period of the fishery during which only preseason conditioning information, denoted symbolically by y_0, is available. Assume now that, during the first and subsequent time periods, additional population data, $y_1, y_2, \ldots$ become successively available. This data may then be used to condition the probability density of N, hopefully in such a manner that our knowledge of the true value of N, as measured by its variance, improves as more data are gathered. At each stage of the fishing season we compute the Bayes risk with respect to the then current probability density of N and adopt a strategy that takes into account all available data and all previous management decision. This will be formalized analytically upon the specification of an appropriate sampling distribution for the $\{y_i\}$ $(i = 1, 2, \ldots m - 1)$. y_m is irrelevant since it is obtained after the final decision δ_m will have been made.

Assume that during each stage of the run some fixed fraction ϵ of the total number of fish entering the fishery is vulnerable to sampling. For example, if the sampling is done by gill nets ϵ may be determined from knowledge of the length, the time of soak, and the efficiency of the net. With such a sampling scheme, it is reasonable to assume that the samples $y_1, y_2, \ldots y_{k-1}$ will have independent Poisson densities with parameters $\lambda_1, \lambda_2, \ldots \lambda_{k-1}$ where $\lambda_i = \epsilon\rho_i N$, i.e., $\epsilon\rho_i N$ is the expected sample size for the ith period and

$$P(Y_i = y_i \mid N) = e^{-\epsilon\rho_i N} \frac{(\epsilon\rho_i N)^{y_i}}{y_i!} \quad (10)$$

where $y_i = 0, 1, \ldots$. Equation (10) and Bayes theorem may be utilized to modify or update Equation (7) to reflect the additional information that is assumed to have become available. Assume that the system is now at the start of the second stage and that the sample y_1 is now available. Bayes theorem gives

$$f_2(N \mid y_0, y_1) = \frac{P(Y_1 = y_1 \mid N) f_1(N \mid y_0)}{\int_0^\infty P(Y_1 = y_1 \mid N') f_1(N' \mid y_0) dN'}. \quad (11)$$

Substituting Equations (7) and (10) in Equation (11) gives, after dividing common factors,

$$f_2(N \mid y_0, y_1) = \frac{N^{\alpha_0 + \nu_1 - 1} e^{-(\beta_0 + \epsilon\rho_1)N}}{\int_0^\infty N^{\alpha_0 + \nu_1 - 1} e^{-(\beta_0 + \epsilon\rho_1)N} dN}. \quad (12)$$

The integral in the denominator of Equation (12) is a standard form expressible in terms of gamma functions which gives

$$f_2(N \mid y_0, y_1) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} N^{\alpha_1 - 1} e^{-\beta_1 N} \quad (13)$$

where $\alpha_1 = \alpha_0 + y_1$ and $\beta_1 = \beta_0 + \epsilon\rho_1$. The updated probability density for N given by Equation (13) is, like the prior density given by Equation (7), a gamma density but with modified parameters α_1 and β_1. The process by which Equation (13) was obtained may be repeated indefinitely to give

$$f_k(N \mid y_0, y_1, \ldots y_{k-1}) = \frac{(\beta_{k-1})^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1})} N^{\alpha_{k-1} - 1} e^{-\beta_{k-1} N} \quad (14)$$

[4]The conditioning of $\hat{\eta}$ by δ only is probably an oversimplification. There is evidence to indicate that $\hat{\eta}$ also depends on the number of fish that enter the fishery during any fishing period.

as the posterior density for N at the start of the kth fishing period. The parameters are given by $\alpha_{k-1} = \alpha_0 + y_1 + y_2 + \ldots + y_{k-1}$ and $\beta_{k-1} = \beta_0 + \epsilon(\rho_2 + \rho_2 + \ldots + \rho_{k-1})$. At this point it is appropriate to observe that, as time progresses and additional population data are obtained, the distribution of N, as specified by the parameters α_{k-1} and β_{k-1}, will more and more reflect the in-season sampling data with a corresponding decrease in the relevance of the preseason information implied by α_0 and β_0.

The probability densities given by Equations (7), (10), and (14) enjoy a peculiar relationship in which the posterior density of N, given by Equation (14), is from the same family as the prior density, Equation (7), for the particular sampling distribution given by Equation (10). Such pairs of densities are called conjugate pairs (DeGroot 1970:159-166). It is obvious that one cannot, in general, be so fortunate as to have parameter and sampling distributions that form a conjugate pair as in the model assumed here. However, DeGroot does outline some somewhat ad hoc procedures for constructing reasonable posterior probability densities.

All of the quantities and distributions necessary to compute the average or expected loss, i.e., the Bayes risk, are now available. The expression for the Bayes risk, to be evaluated at the start of the kth fishing period, may be written formally as

$$R_k(\delta_1, \delta_2, \ldots \delta_m \mid y_0, y_1, \ldots y_{k-1}) = \int_0^\infty f_k(N \mid y_0, y_1, \ldots y_{k-1})dN \int_0^1 d\hat{\eta}_1 \ldots \int_0^1 d\hat{\eta}_m L(E_n) \prod_{i=1}^{m} g_i(\hat{\eta}_i \mid \delta_i) \quad (15)$$

where g_i, $L(E_n)$, and f_k are given by Equations (9), (6), and (14) respectively. Notice that the Bayes risk as given by Equation (15) is a function not only of the decisions already made, $\delta_1, \delta_2, \ldots \delta_{k-1}$ and the decision about to be made, δ_k, but of all future decisions $\delta_{k+1}, \ldots \delta_m$ as well. This dependence on all decisions, past, present, and future, reflects the assumption that the loss is a function primarily of the final state of the system, i.e., to a first approximation one cannot ascribe values to individual units of escapement during the season but only to the final total escapement. This presents no particular analytical difficulties since any particular sequence of optimum future decisions $\delta_{k+1}, \ldots \delta_m$ is certainly subject to revision as time passes and additional information becomes available.

Substituting Equations (6), (9), and (14) in Equation (15) gives

$$R_k(\delta_1, \delta_2, \ldots \delta_m \mid y_0, y_1, \ldots y_{k-1}) = \frac{(\beta_{k-1})^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1})} \int_0^\infty N^{\alpha_{k-1}-1} e^{-\beta_{k-1}N} dN \int_0^1 d\hat{\eta}_1 \ldots \int_0^1 d\hat{\eta}_m \left\{ L'(E_0, E_{n-1}, \ldots E_{n-K}) - (aE_n e^{-bE_n} - E_n) \right\} \cdot \prod_{j=1}^{m} \frac{\Gamma(\nu_j + \mu_j)}{\Gamma(\nu_j)\Gamma(\mu_j)} \hat{\eta}_j^{\nu_j - 1} (1 - \hat{\eta}_j)^{\mu_j - 1} \quad (16)$$

where $L'(E_0, E_{n-1}, E_{n-2}, \ldots E_{n-K})$ denotes that portion of the loss function that does not depend on E_n. Thus L' is a fixed quantity and may be removed from the integral signs. This leaves only probability densities, which must integrate out to unity, so that

$$R_k(\delta_1, \delta_2, \ldots \delta_m \mid y_0, y_1, \ldots y_{k-1}) = L'(E_0, E_{n-1}, \ldots E_{n-K}) - \frac{(\beta_{k-1})^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1})} \int_0^\infty N^{\alpha_{k-1}-1} e^{-\beta_{k-1}N} dN \int_0^1 d\hat{\eta}_1 \ldots \int_0^1 d\hat{\eta}_m \left(ae^{-bN\sum_{i=1}^{m} \rho_i \hat{\eta}_i} - 1\right)N \sum_{i=1}^{m} \rho_i \hat{\eta}_i \cdot \prod_{j=1}^{m} \frac{\Gamma(\nu_j + \mu_j)}{\Gamma(\nu_j)\Gamma(\mu_j)} \hat{\eta}_j^{\nu_j - 1} (1 - \hat{\eta}_j)^{\mu_j - 1} \quad (17)$$

where the escapement E_n has been expressed as

$$E_n = N \sum_{i=1}^{m} \rho_i \hat{\eta}_i .$$

The integrations in Equation (17) cannot be performed as expressed. If the order of the integrations is reversed, the integration with respect to N may be performed but the remaining integrations over $\hat{\eta}_1, \hat{\eta}_2, \ldots \hat{\eta}_m$ will be virtually impossible. However, if the exponential term $\exp\left(-bN \sum_{i=1}^{m} \rho_i \hat{\eta}_i\right)$ is expanded in its Maclaurin series and if the resulting multinomials of the form $\frac{1}{n!}\left(-bN \sum_{i=1}^{m} \rho_i \hat{\eta}_i\right)^n$ $(n = 0, 1, \ldots)$ are expanded according to the multinomial theorem, the integrand in Equation (17) will be in a completely factored form. As a result of this factorization, the integrals take the form of various moments about the origin. These integrals are all standard forms (c.f., Bierens de Haan 1939). The reader will be spared the details of this reduction and the ensuing integrations. The final expression for the Bayes risk is

$$R_k\,(\delta_1,\delta_2,\ldots\delta_m\,|\,y_0,y_1,\ldots y_{k-1})$$

$$= L'(E_0,E_{n-1},\ldots E_{n-K}) + \frac{\alpha_{k-1}}{\beta_{k-1}}\sum_{i=1}^{m}\frac{\rho_i\nu_i}{\nu_i+\mu_i}$$

$$-\frac{a}{\beta_{k-1}}\sum_{n=0}^{\infty}\left(-\frac{b}{\beta_{k-1}}\right)^n\frac{\Gamma(\alpha_{k-1}+n-1)}{n!\,\Gamma(\alpha_{k-1})}$$

$$\sum_{\substack{k_i=0\\ \Sigma k_i=n}}^{n}\binom{n}{k_1k_2\ldots k_m}\sum_{i=1}^{m}\frac{\rho_i(\nu_i+k_i)}{\nu_i+\mu_i+k_i}$$

$$\cdot\prod_{j=1}^{m}\frac{\Gamma(\nu_j+\mu_j)\,\Gamma(\nu_j+k_j)}{\Gamma(\nu_j)\,\Gamma(\nu_j+\mu_j+k_j)}\,\rho_j^{k_j} \qquad (18)$$

where $\binom{n}{k_1k_2\ldots k_m}$ denotes a multinomial coefficient.

A slightly different form for the risk may be obtained under an alternate set of assumptions. Considerable emphasis has heretofore been placed on the conjugacy of the gamma-Poisson families of distributions. The gamma-Poisson assumption is a reasonable one and the resulting conjugacy lends a certain elegance. However, this line of analysis results in posterior gamma parameters (α_k, β_k) that, among other things, depend on the run fractions $\{\rho_i\}$ $(i = 1, \ldots k)$. This parameter dependence on the run fractions virtually precludes treating the set $\{\rho_i\}$ $(i = 1, \ldots k)$ as anything but fixed quantities; i.e., once a variable becomes the argument of a gamma function one has usually arrived at an analytical dead end. In actual practice, however, the quantities $\{\rho_i\}$ $(i = 1, \ldots m)$ are random variables since there may be considerable year-to-year variation in the time profile of the run. Such temporal variation may be of considerable importance in Bristol Bay because of the large magnitude of the run and its short duration.

It has been suggested (O. A. Mathisen, pers. commun. and others) that the probability density of N is most appropriately conditional upon the catch-per-unit-effort (CPUE) observed during the course of the run. In so doing one can remove the explicit dependence of (α_k, β_k) on $\{\rho_i\}$ $(i = 1, \ldots k)$. An implicit dependence remains, however, since the CPUE will be a function of the run fractions. One can formally bypass this dependence, however, by relating the density of N directly to the CPUE. In so doing one can then introduce temporal variability in the set $\{\rho_i\}$ $(i = 1, \ldots m)$ and in evaluating the Bayes risk an additional expectation with respect to the density of these random variables must be taken.

An almost ideal probability density to describe the run fractions is the Dirichlet density defined by

$$h(\rho_1,\rho_2,\ldots\rho_m) = \frac{\Gamma(\gamma_1+\gamma_2+\ldots+\gamma_m)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\ldots\Gamma(\gamma_m)}\,\rho_1^{\gamma_1-1}\rho_2^{\gamma_2-1}\ldots\rho_m^{\gamma_m-1} \qquad (19)$$

where $\rho_i \geq 0$ for all i. As written this density is singular since the variates must satisfy the side condition $\sum_{i=1}^{m}\rho_i = 1$. The choice of the parameters $(\gamma_1, \gamma_2, \ldots \gamma_m)$ then permits the specification of any m of the means, variances, and covariances of the $\{\rho_i\}$ $(i = 1, \ldots m)$. If Equation (19) is substituted in Equation (16) the integrations with respect to N and $(\hat{\eta}_1, \hat{\eta}_2, \ldots \hat{\eta}_m)$ may be done as before. The remaining integrals over $(\rho_1, \rho_2, \ldots \rho_m)$ are all Dirichlet integrals (Wilks 1962:177, et seq.) for which the values are readily determined. The resulting Bayes risk for this case may then be shown to be given by

$$R_k(\delta_1,\delta_2,\ldots\delta_m\,|\,\text{CPUE})$$

$$= L'(E_0,E_{n-1},\ldots E_{n-K}) + \frac{a\alpha_{k-1}}{G\beta_{k-1}}\sum_{i=1}^{m}\frac{\nu_i\gamma_i}{\nu_i+\mu_i}$$

$$-\frac{a}{\beta_{k-1}}\sum_{n=0}^{\infty}\left(-\frac{b}{\beta_{k-1}}\right)^n\frac{\Gamma(\alpha_{k-1}+n+1)}{\Gamma(\alpha_{k-1})\,\Gamma(G+n+1)}$$

$$\sum_{\substack{k_i=0\\ \Sigma k_i=n}}^{n}\binom{n}{k_1k_2\ldots k_m}$$

$$\cdot\prod_{i=1}^{m}\frac{\Gamma(\nu_i+k_i)\,\Gamma(\nu_i+\mu_i)\,\Gamma(\gamma_i+k_i)}{\Gamma(\nu_i)\,\Gamma(\nu_i+\mu_i+k_i)\,\Gamma(\gamma_i)}$$

$$\sum_{j=1}^{m}\frac{(\gamma_j+k_j)(\nu_j+k_j)}{\nu_j+\mu_j+k_j} \qquad (20)$$

where $G = \sum_{i=1}^{m}\gamma_i$.

Equations (18) and (20) are somewhat intimidating, particularly if one were to attempt to infer the qualitative behavior of the system as the parameters descriptive of the fishery and its management are varied. Indeed, Equations (18) and (20) are virtually useless for this purpose with the exception of the determination of certain limiting behavior as the appropriate parameters assume their extreme values. However, Equations (18) and (20) do have the virtue that, in closed form, the most crucial features of the fishery dynamics and statistics are accommodated in a quantitative and, hopefully, reasonably accurate fashion.

A NUMERICAL EXAMPLE

The foregoing mathematical model was applied to the simulated management of the Wood River system of Bristol Bay. It should be emphasized at the outset, however, that the assumptions, methods, and results presented here should in no way be construed as representing a management scheme preferable to those currently in use. The Wood River was chosen simply because, based on Mathews' (1966) data, it seemed to follow the Ricker spawner-return curve reasonably well.

In the example considered here, the model was limited to a fishing season of five time periods during each of which a choice of two management decisions was possible. This limitation was necessary to avoid inordinately lengthy calculations. Ricker parameter values of $a = 4.077$ and $b = 0.8 \times 10^{-6}$, which were used by Mathews, were used here. The return was assumed to consist of only the progeny of a single spawning group K years prior where K is arbitrary, i.e.,

$$p_i = \begin{cases} 1 & i = K \\ 0 & i \neq K \end{cases}.$$

All prior escapements were assumed to be the optimum escapement E_0 so that the loss function given by Equation (5) becomes

$$L_n = (aE_0e^{-bE_0} - E_0) - (aE_ne^{-bE_n} - E_n).$$

For the above values of the Ricker parameters, the MSY escapement is given by $E_0 = 709{,}000$. The expected value and standard deviation of a $\Gamma(\alpha_0, \beta_0)$ variate are given by α_0/β_0 and $\alpha_0^{1/2}/\beta_0$, respectively. In terms of the Ricker parameters, the expected run size is given by $aE_0 \exp(-bE_0)$ which determines the ratio $\alpha_0/\beta_0 = 1.64 \times 10^6$. An initial (i.e., preseason) standard deviation of one-half the expected run size was assumed. In terms of the gamma parameters this gives $\alpha_0^{1/2}/\beta_0 = \alpha_0/2\beta_0$ or $\alpha_0 = 4.0$ and $\beta_0 = 2.44 \times 10^{-6}$. The two management strategies assumed were complete closure (option 2) and one level of opening (option 1). In terms of the beta parameters, closure is simulated merely by setting $\mu_2 = 0$ with an arbitrary positive value for ν_2. During fishery opening it was assumed that an average of 80% of the available fish are caught with a standard deviation of 0.25. This gives $(\nu_1, \mu_1) = (0.312, 1.248)$ as the appropriate beta parameters. The set of run fractions $\{\rho_i\}$ $(i = 1, \ldots 5)$ was determined from the time profile proposed by Royce (1965). Values of 0.156, 0.282, 0.348, 0.160, and 0.054, using five equal length time intervals, were obtained. No attempt was made to treat the run fractions as random variables. All of the parameter values were chosen to reflect reasonably well the known behavior of the system.

The fishery dynamics were treated by two distinct methods. The first method utilized the gamma prior density for N with a Poisson sampling density thus, through conjugacy, giving a gamma posterior density. A gamma posterior distribution was also assumed in the second method but the posterior gamma parameters were back-calculated after introducing prescribed stage-to-stage trends in the population mean and standard deviation.

The Bayes risk at each stage was computed for each of the $2^5 = 32$ total possible sequences of decisions, past, present, and future; i.e., no attempt was made to formulate and solve the functional equation associated with dynamic programming.[5] While relatively unsophisticated, this approach does permit one to use hindsight to determine, ex post facto, what an optimum previous strategy would have been, given the information currently available. In real life, of course, "what might have been" is irrelevant in the management of a dynamic system—one must optimize the system as it exists in real time in accordance with the principal of optimality, the relevant homily for which might well be "what's past is prologue."

The numerical results are summarized in Tables 1 to 3. Tables 1 and 2 give the optimum strategies and corresponding minimum Bayes risks for a gamma prior run size distribution with simulated

[5]Subsequent to the submission of this paper, C. J. Walters (1975) published a paper in which the ideas of dynamic programming were applied to the optimum year to year management of a salmon fishery. His work is of considerable interest, particularly since he managed to impose the principle of optimality and carry out the backward recursive scheme proposed by Bellman (1957). It remains to be seen if this method can be applied to the decision theoretic model presented here, but I am no longer as pessimistic as I formerly was.

TABLE 1.—Optimum strategies and minimum Bayes risks for a five-period, two-decision fishery with a sampling fraction $\epsilon = 1 \times 10^{-3}$.

Time period (i)	1	2	3	4	5
Run fraction (ρ_i)	0.156	0.282	0.348	0.160	0.054
Poisson parameter (λ_i)	256	462	570	262	—
$y_i = \lambda_i$					
Simulated samples (y_i)	256	462	570	262	—
Optimum strategy	open	open	close	open	open
Minimum Bayes risk	1.80×10^5	3.49×10^4	3.33×10^4	3.29×10^4	3.28×10^4
$y_i = 2\lambda_i$					
Simulated samples (y_i)	512	924	1,140	564	—
Optimum strategy	open	open	open	open	close
Minimum Bayes risk	1.80×10^5	1.89×10^5	1.70×10^5	1.90×10^5	1.90×10^5
$y_i = \frac{1}{2}\lambda_i$					
Simulated samples (y_i)	128	231	285	131	—
Optimum strategy	open	close	close	close	close
Minimum Bayes risk	1.80×10^5	6.90×10^3	3.35×10^3	2.48×10^3	2.29×10^3

TABLE 2.—Optimum strategies and minimum Bayes risks for a five-period, two-decision fishery with a sampling fraction $\epsilon = 1 \times 10^{-4}$.

Time period (i)	1	2	3	4	5
Run fraction (ρ_i)	0.156	0.282	0.348	0.160	0.054
Poisson parameter (λ_i)	26	46	57	26	—
$y_i = \lambda_i$					
Simulated samples (y_i)	26	46	57	26	—
Optimum strategy	open	close	open	open	open
Minimum Bayes risk	1.80×10^5	5.86×10^4	4.55×10^4	4.19×10^4	4.11×10^4
$y_i = 2\lambda_i$					
Simulated samples (y_i)	52	92	114	52	—
Optimum strategy	open	open	open	open	close
Minimum Bayes risk	1.80×10^5	1.86×10^5	1.87×10^5	1.89×10^5	1.88×10^5
$y_i = \frac{1}{2}\lambda_i$					
Simulated samples (y_i)	13	23	29	13	—
Optimum strategy	open	close	close	close	open
Minimum Bayes risk	1.80×10^5	4.14×10^4	1.85×10^4	1.12×10^4	9.65×10^3

TABLE 3.—Optimum strategies and minimum Bayes risks for a five-period, two-decision fishery with linear stage-to-stage trends in the expected run size and the run size standard deviation with preseason parameters $\alpha_0 = 4.0$ and $\beta_0 = 2.44 \times 10^{-6}$.

Time period (i)	1	2	3	4	5
Run fraction (ρ_i)	0.156	0.282	0.348	0.160	0.054
Constant expected run size:					
α_{i-1}/β_{i-1}	1.64×10^6	1.64×10^6	1.64×10^6	1.64×10^6	1.64×10^6
$\sqrt{\alpha_{i-1}}/\beta_{i-1}$	8.20×10^5	6.89×10^5	5.57×10^5	4.26×10^5	2.95×10^5
Optimum strategy	open	close	open	open	open
Minimum Bayes risk	1.80×10^5	1.41×10^5	1.07×10^5	7.87×10^4	5.74×10^4
Increasing expected run size:					
α_{i-1}/β_{i-1}	1.64×10^6	1.97×10^7	2.30×10^6	2.63×10^6	2.95×10^6
$\sqrt{\alpha_{i-1}}/\beta_{i-1}$	8.20×10^5	6.89×10^5	5.57×10^5	4.26×10^5	2.95×10^5
Optimum strategy	open	open	open	close	open
Minimum Bayes risk	1.80×10^5	1.40×10^5	1.26×10^5	1.43×10^5	1.90×10^5
Decreasing expected run size:					
α_{i-1}/β_{i-1}	1.64×10^6	1.48×10^6	1.31×10^6	1.15×10^6	9.84×10^5
$\sqrt{\alpha_{i-1}}/\beta_{i-1}$	8.20×10^5	6.89×10^5	5.57×10^5	4.26×10^5	2.95×10^5
Optimum strategy	open	open	close	close	close
Minimum Bayes risk	1.80×10^5	1.52×10^5	1.25×10^5	9.57×10^4	6.68×10^4

Poisson sampling. The sampling was intended to simulate actual run sizes equal to, greater than, or less than the preseason estimate of the run size, α_0/β_0. The Poisson sampling was done by brute force in which sample values exactly equal to the desired expected values were chosen. For example, to simulate an actual run size twice that based on the preseason parameters we choose $y_i = 2\lambda_i$ where $\lambda_i = \epsilon\rho_i\ \alpha_0/\beta_0$ is the Poisson parameter for the ith period obtained from the preseason parameters. The deterministic samples (which is really a contradiction in terms) permit one to elicit the response of the system to specified input stimuli.

The Bayes risks are all in units of numbers of fish. The optimum strategy is that strategy which minimizes the Bayes risk given that all prior decisions were optimum for the time periods in which they were made. In other words, the "hindsight" feature was not utilized to "improve" a past decision—once made any decision is retained through all subsequent stages.

The mathematical machinery developed generally gives intuitively reasonable results. Specifically, the tendency toward larger or smaller run sizes results in optimum strategies that tend successively toward more or fewer open periods respectively. The Bayes risk generally, but not always, decreases as the season progresses, largely reflecting the decreasing variances in the estimates of the run size. Increases in the Bayes risk can usually be attributed to past decisions that, in the light of subsequent sampling, are no longer optimum thus requiring corrective action.

CONCLUSIONS

The mathematical models assumed and developed here for the objective management of a typical salmon fishery, as previously noted, are based on quite specific functional forms and thus represent somewhat of an idealized situation. However, these functions were chosen to reflect the behavior of the system insofar as the knowledge of such behavior is available. Indeed, the acquisition of such detailed knowledge is an important area of current research and subsequent refinements of the statistics will be possible as more data are gathered.

Of more concern than the accuracy of the fine-scale mathematical behavior of the system is the appropriateness of the basic mathematical theory upon which the models are built. I feel that statistical decision theory is a most natural framework on which to base an objective management model. The nomenclature lends support to this view. For example, the equivalence of a management decision and a statistical decision is obvious.[6] The term risk, in the economic if not the strict Bayesian sense, is frequently used in discussions of fishery management. Finally, Bayes theorem provides a convenient and theoretically appropriate method for accommodating the combined data acquisition and dynamics of the fishery.

[6]This equivalence is not always evident even within decision theory itself. For example, it requires a slight mental contortion to treat statistical estimation as an application of decision theory as the statisticians have done.

Advantage has been taken of some powerful analytical tools to characterize salmon fishery management. However, any enthusiasm for these quite contemporary methods should be tempered somewhat by consideration of some of the specific practical difficulties likely to be encountered. One of these, mentioned in Lord (1973), is the difficulty associated with multistage dynamic processes. While the fishery management problem under discussion falls very naturally into a class of stochastic dynamic programs it is not yet obvious whether the functional equation arising from the imposition of the principal of optimality can be formulated or solved in a useful fashion. The calculations done here were more of the brute force variety in which all strategy combinations, optimal or not, were considered. In other words, the backward recurrence scheme central to dynamic programming was not used to reduce the total number of possible strategies to be considered. In so doing, the "Curse of Dimensionality," about which Bellman (1957:6) so aptly warned, proved to be a limiting condition. To evaluate completely the five-stage, two-decision fishery considered here required from 10 to 15 min of Control Data Corporation[7] 6400 central processor time for each set of input parameters. This is not a trivial numerical effort and should give one pause when considering more elaborate models.

In conclusion I feel that advantage should be taken of the appropriate analytical tools as they are made available by the mathematicians or, at the very least, such tools should be investigated. However, the availability of such methods in no way indicates their eventual practicality for any specific problem. For this careful additional investigation is necessary.

LITERATURE CITED

BELLMAN, R.
1957. Dynamic programming. Princeton Univ. Press, 340 p.

BIERENS DE HAAN, D.
1939. Nouvelles tables d'intégrules définies. G. E. Stechert & Co., 716 p.

DEGROOT, M. H.
1970. Optimal statistical decisions. McGraw-Hill Book Co., Inc., 489 p.

LORD, G. E.
1973. Characterization of the optimum data acquisition and management of a salmon fishery as a stochastic dynamic program. Fish. Bull., U.S. 71:1029-1037.

[7]Reference to trade names does not imply endorsement by the National Marine Fisheries Service, NOAA.

MATHEWS, S. B.
1966. The economic consequences of forecasting sockeye salmon (*Oncorhynchus nerka*, Walbaum) runs to Bristol Bay, Alaska: A computer simulation study of the potential benefits to a salmon canning industry from accurate forecasts of the runs. Ph.D. Thesis, Univ. Washington, Seattle, 238 p.

ROTHSCHILD, B. J., AND J. W. BALSIGER.
1971. A linear-programming solution to salmon management. Fish. Bull., U.S. 69:117-140.

ROYCE, W. F.
1965. Almanac of Bristol Bay sockeye salmon. Univ. Wash. Fish. Res. Inst. Circ. 235, 48 p.

WALTERS, C. J.
1975. Optimal harvest strategies for salmon in relation to environmental variability and uncertain production parameters. J. Fish. Res. Board Can. 32:1777-1784.

WILKS, S. S.
1962. Mathematical statistics. John Wiley & Sons, Inc., N.Y., 644 p.