FISHERY REGULATION VIA OPTIMAL CONTROL THEORY[1]

WILLIAM J. PALM[2]

ABSTRACT

This paper attempts to show how control theory can be used to formulate a regulatory scheme for fisheries. The regulatory mechanism considered is a limit imposed on fishing effort. It is shown that static optimization methods, such as maximum equilibrium yield analysis, need to be supplemented with dynamic methods, such as optimal control theory, which take into account the variable nature of a fishery. The dynamic analysis is used to show that the size of a limit on effort should be a feedback function of the variables in the state of the fishery. The concept of the Linear-Quadratic Optimal Control Problem is introduced as a method for devising such a feedback scheme for fishery regulation.

A single-variable logistic model is used to introduce the basic concepts. A model with three variables is then analyzed to show how the techniques are easily extended to the general multivariable case. Details of the general method are given in an Appendix.

The need for fishery regulation is apparent and will become even more important with the establishment of resource management zones off our coasts. Regulatory mechanisms include catch quotas and limits on fishing effort (number of boats permitted entry into the fishery, number of hooks used, etc.). A mathematical model of the fishery, which includes biological and perhaps economic factors, is useful for determining the best regulatory scheme. Some of the more familiar examples of these models are given by Schaefer (1954, 1968), Beverton and Holt (1957), Ricker (1958), Larkin (1963, 1966), Pella and Tomlinson (1969) and Fox (1970). The above models are said to be dynamic because they utilize differential equations to describe how the fishery changes with time. The inclusion of economic factors, multiple species, and other biological variables, such as size and age, results in multivariable models which are quite complex.

Much of the analysis of fisheries is based on the concept of an equilibrium. Perhaps the best known is the maximum equilibrium yield analysis. However, equilibrium is an idealization and is never actually encountered in reality because continually changing environmental influences act as disturbances which displace the system from its equilibrium condition. For unstable systems this is disastrous because equilibrium is never regained, and for stable systems with large time constants, the return to equilibrium might take so long as to negate the assumptions and usefulness of the equilibrium-based analysis. Thus "static" or equilibrium-based analysis should be supplemented with dynamic methods which take into account the variable nature of the fishery. A purpose of this paper is to show that the above considerations indicate that any regulatory scheme should contain "feedback"; that is, the size of any quota or limit should be a function of the state of the fishery. Also, the concept of the Linear-Quadratic Optimal Control Problem will be introduced as one way of devising such a feedback scheme for fishery regulation.

The Linear-Quadratic Optimal Control Problem, which has been widely applied in engineering, is one method within the larger framework of optimal control theory. Other optimal control methods have recently been applied to problems in fishery management which are unlike the problem treated here. Goh (1969, 1973) applied the so-called "singular" control method to the problem of maximizing yield with a single-species model. Saila (in press) describes Goh's results in more detail. Clark et al. (1973) analyze the problem of optimal reduction of effort in an overexploited fishery. They calculate the fishing mortality function which maximizes the total present value of all profits and utilize a Beverton-Holt model for the fishery. Clark (1973) has presented a similar analysis for a logistic fishery model. The above three analyses lead to control functions which have been loosely described as a "bang-bang" control

[1]Part of this work is a result of research sponsored by NOAA, Office of Sea Grant, Department of Commerce under Grant #98-20-6807C.

[2]Department of Mechanical Engineering and The Institute of Environmental Biology, University of Rhode Island, Kingston, RI 02881.

Manuscript accepted January 1974.
FISHERY BULLETIN: VOL. 73, NO. 4, 1975.

because the optimal values of the control variable lie at its boundaries. Thus the control variable switches between a lower value (usually zero) and an upper value which might be difficult to specify.

There are advantages as well as limitations with the linear-quadratic approach as compared with the bang-bang control approach. With the linear-quadratic approach, quantities such as yield and present value of profits are not directly maximized to obtain the feedback control function, as is done with the bang-bang approach. Rather, the maximization is first done with static methods, and then a feedback control function is constructed to keep the system near the resulting equilibrium condition. To do this, the system equations are linearized about the equilibrium. If disturbances carry the system far from equilibrium, the linearization breaks down. However, this is generally not a serious limitation, since the feedback control function is designed to counteract disturbances and to keep the system near equilibrium. The method is not restricted to equilibrium analysis, and frequently the two approaches are combined by using bang-bang control methods, instead of static methods, to compute an optimal "open-loop" control function. Linearization of the model around the resulting trajectory enables the linear-quadratic method to be used to synthesize a closed-loop (feedback) control function to keep the system on the optimal trajectory (Ho and Bryson 1969).

A significant advantage of the linear-quadratic approach is that it allows the use of linear control theory, whose techniques are more highly developed and easier to apply than the nonlinear techniques required for bang-bang control analysis. Powerful methods of compensating for incomplete information, uncertainties in measurements, model parameters, and model structure are available for the linear-quadratic approach but are scarce for the bang-bang control approach. Also, solutions to bang-bang control problems are extremely difficult to obtain if the model contains more than two variables.

First a single-variable model is used to illustrate the basic concepts. A model with three variables is then analyzed to show how the techniques are easily extended to the general multivariable case. The details of the general method are in the Appendix. There it is also shown in more detail why static optimization methods, such as linear programming, and dynamic optimization methods, such as optimal control theory, should not be treated as competing methods but rather should be used together as part of the total approach to the problem because they are mutually complementary methods. This is mentioned because there is a tendency among economics-oriented analysts to use static methods, whereas analysts with control-theory backgrounds tend toward dynamic methods.

It is assumed that the reader is familiar with the fundamentals of differential equations and matrix operations. A matrix will be denoted by brackets [ ]; a matrix transpose by $[\ ]^T$; and a column vector by a bar underneath, as $\underline{x}$.

SINGLE-VARIABLE MODEL

The following model is the Schaefer or logistic model:

$$\frac{dN}{dt} = aN - bN^2 - qfN, \quad (1)$$

where N is the biomass or number of catchable fish in the fishery, t is time, q is the catchability coefficient, and f is the fishing effort. The constant a is the intrinsic rate of natural increase of the population, and the constant b is related to the carrying capacity of the environment c by the relation: $b = a/c$. The system's equilibrium (N_{eq}, f_{eq}) is found by setting the derivative in Equation (1) equal to zero:

$$0 = aN_{eq} - bN^2_{eq} - qf_{eq}N_{eq}.$$

The equilibrium yield Y_{eq} is:

$$Y_{eq} = qf_{eq}N_{eq} = (a - bN_{eq})\,N_{eq}.$$

To find the maximum equilibrium yield, we differentiate Y_{eq} with respect to N_{eq}, and set this result equal to zero.

$$\frac{dY_{eq}}{dN_{eq}} = a - 2bN_{eq} = 0.$$

Solving this for the population size and fishing effort corresponding to maximum equilibrium yield, we obtain:

$$N_{eq} = a/2b,$$

$$f_{eq} = \frac{(a - bN_{eq})\,N_{eq}}{q\,N_{eq}} = \frac{a}{2q}.$$

Note that a static optimization method, calculus, has been used to find an optimal equilibrium. To analyze the model's behavior in the vicinity of the equilibrium point, Equation (1) is linearized by expanding the right-hand side in a two-variable Taylor series about the point (N_{eq}, f_{eq}), and keeping only the first-order terms (see Appendix). This gives:

$$\frac{dx}{dt} = \left(\frac{\delta g}{\delta N}\right)_{eq} x + \left(\frac{\delta g}{\delta f}\right)_{eq} u$$

where: $g = aN - bN^2 - qfN$

$$x = N - N_{eq} \qquad (2)$$

$$u = f - f_{eq}. \qquad (3)$$

The new variables x and u are the deviations in population density and fishing effort from their equilibrium values. After evaluating the derivatives of g at the equilibrium point, we obtain:

$$\frac{dx}{dt} = -\frac{a}{2}x - \frac{aq}{2b}u. \qquad (4)$$

If the fishing effort is kept constant at its equilibrium value, then $u = 0$ and

$$\frac{dx}{dt} = -\frac{a}{2}x.$$

This system is stable for all positive values of a, which means that if disturbed from equilibrium, the population will eventually return to it. The solution is:

$$x(t) = x(t_o)\, e^{-\frac{a}{2}(t - t_o)},$$

where $x(t_o)$ is the deviation at time t_o. Since the time constant for this system is $2/a$, it will take that amount of time for the deviation to decay by 63% and for four time constants to decay by 98%. If the constant a is small, this time can be very large. Also, by keeping the fishing effort constant, we cannot take advantage of higher yields obtainable when $x(t_o)>0$, and risk overexploiting when $x(t_o)<0$. We will show that by making the fishing effort a function of population level, we can change the system's time constant and also avoid the above difficulties.

For example, the results of Schaefer (1954) give the following values for the Pacific halibut:

$$a = 0.67$$
$$b = 3.05 \times 10^{-9}$$
$$q = 3.95 \times 10^{-5}$$

where N is in pounds, t in years, and f in number of skates. A standard skate of halibut gear consists of eight lines of 300 feet each in length, with shorter lines with hooks attached at 10-foot intervals (Carrothers 1941). The time constant is 3 yr, and thus 12 yr are required for a deviation in population to disappear, assuming no other disturbances act during that time.

If we now specify, by means of a "performance index," that we wish to keep N near N_{eq} while minimizing the variation in f required to do so, we can design a regulatory procedure which will keep the fishery near the maximum equilibrium yield condition. The performance index J which specifies this desire is the so-called quadratic index:

$$J = \int_0^\infty (Qx^2 + Ru^2)\, dt.$$

The squared terms indicate that we make no distinction between positive and negative deviations from equilibrium. The positive constants Q and R are the weighting factors which indicate the relative importance placed on keeping N near N_{eq} (x near 0) versus keeping f near f_{eq} (u near 0). The infinite upper limit indicates that we are interested in long-term as well as short-term effects of our fishing effort regulation.

The problem of determining the function u, which minimizes the performance index, is solved by the application of optimal control theory. Since the system, Equation (4), is linear, and the index is quadratic, the problem formulated above is referred to as the Linear-Quadratic Optimal Control Problem.

The solution for the control function is (see Appendix):

$$u = -Kx \qquad (5)$$

$$K = -\frac{1}{R}\frac{aq}{2b}P$$

where P is the positive steady-state solution of the so-called Riccati equation:

$$\frac{dP}{dt} = -aP - \frac{1}{R}\left(\frac{aq}{2b}\right)^2 P^2 + Q$$

with initial condition $P(0) = 0$. The solution for P is:

$$P = \frac{-aR + a\sqrt{R^2 + q^2a^2RQ/b^2}}{q^2a^2/2b^2}$$

Thus P and K are functions of the weighting factors R and Q, which must be specified.

Note that this method yields three results: 1) that the optimal control function for u is a linear function of x (a linear feedback law); 2) the means to calculate the feedback gain K, once R and Q are specified; and 3) that K is negative in this example (we assume that a, b, and q are positive). The third result indicates that the control law, Equation (5), opportunely calls for an increase in fishing effort when the population increases ($x > 0$), and conservatively calls for a decrease in effort when the population decreases ($x < 0$).

In this simple single-variable case we can utilize the first result and avoid specifying R and Q by substituting u from Equation (5) into Equation (4). The result is:

$$\frac{dx}{dt} = \left(\frac{qa}{2b}K - \frac{a}{2}\right)x.$$

The time constant for this system is:

$$\tau = \frac{1}{\frac{a}{2} - \frac{qa}{2b}K}. \qquad (6)$$

Using this approach it is possible to choose K so as to give a desired value of the time constant.

Alternately, K may be chosen by specifying the magnitude of the deviation in fishing effort we will allow in order to counteract an expected deviation in population level. Written in terms of magnitudes, Equation (5) becomes:

$$K = -\frac{u_m}{x_m}$$

where: x_m = maximum magnitude expected for x
u_m = maximum magnitude specified for u.

Once K has been determined, f as a function of N can be found by substituting x and u from Equations (2) and (3) into Equation (5) to obtain:

$$f = f_{eq} - K(N - N_{eq}). \qquad (7)$$

To evaluate the effects of the above regulation scheme under various conditions, the above expression is substituted into Equation (1), which can then be solved by computer for N and f as functions of time.

As an example with the previously mentioned results of Schaefer (1954) for the Pacific halibut, a maximum deviation in N of 5% from N_{eq} was postulated, and a maximum deviation in f of 5% from f_{eq} was specified. Thus:

$$N_{eq} = a/2b = 1.098 \times 10^8$$
$$f_{eq} = a/2q = 8.48 \times 10^3$$
$$x_m = 0.05N_{eq}$$
$$u_m = 0.05f_{eq}.$$

Using the second method for computing K, we obtain:

$$K = -\frac{0.05f_{eq}}{0.05N_{eq}} = -0.772 \times 10^{-4}.$$

From Equation (6) the new time constant is found to be 1.5 yr, which is one-half the value for the case without feedback control. The fishing effort found from Equation (7) is:

$$f = \frac{a}{2q} + \frac{b}{q}\left(N - \frac{a}{2b}\right) = \frac{b}{q}N = 0.772 \times 10^{-4}N. \quad (8)$$

In view of the impossibility of continuously and instantly measuring population size and varying fishing effort, f as given by Equation (8) was interpreted as follows. It was assumed that a limit is imposed on fishing effort at the beginning of each year and held constant during that year, and its value f is calculated from Equation (8), with N being the average population over a yearly interval terminating three-tenths of a year before the imposition of the new limit. That is, three-tenths of a year is allowed for collecting and analyzing the population data used to calculate the next year's limit. With this discretized version of f, computer simulation results show that the system time constant is 1.8 yr, which is reasonably close to the 1.5 yr predicted by the continuous model. Thus it is possible to use the analysis based on the continuous model in the realistic situation involving data-collection limitations and limit-imposition constraints.

THREE-VARIABLE EXAMPLE

An advantage of the optimal control method is its ability to accommodate multivariable system models such as multispecies models; models describing economic as well as biological phenomena; and detailed population models incorporating size, age, temperature, food supply, etc. Once a three-variable example is presented, generalization of the technique to models with more than three variables is straightforward. The following model of a single species population was developed by Timin and Collier (1971) and contains three state variables: N, the population density; W, the mean biomass per organism; and E, the food density. The model is given in dimensionless form, and thus the values of the model variables are relative to reference values. The system's dynamics are described by the following equations:

$$\frac{dN}{dt} = (b-d)N - f \tag{9}$$

$$\frac{dE}{dt} = \sigma - qN - \theta E \tag{10}$$

$$\frac{dW}{dt} = gq - (W+c)b - \mu W - \frac{(W_h - W)f}{N} \tag{11}$$

where: t = time measured in a dimensionless unit equal to the time required for the organism to metabolize an amount of food equal to its own dry weight (usually between two and four weeks for commercial fish species)

b, d = birth and death rates per individual

f = fishing rate

g = the ratio of the quantity (energy ingested minus energy not assimilated, minus energy expended to catch, ingest and assimilate the ingested food) to the amount of energy ingested

q = food ingestion rate per individual

c = coefficient of energy loss associated with births

μ = metabolic heat loss coefficient

σ = rate of food supply

θ = proportionality constant for the rate of food leaving the system through decay or flushing

W_h = mean organism biomass of harvested individuals.

Functional forms and parameters given as typical by Timin and Collier are:

$b = 3.8W^2 - 3.8W + 0.95$ $\qquad \theta = 0.1$

$d = 0.19/(2W-1)$ $\qquad g = 0.2$

$\sigma = 3$ $\qquad c = 0.05$

$q = \dfrac{W^{2/3}E}{1 + 0.1E}$ $\qquad \mu = W^{-1/3}.$

Static optimization can be used to determine the maximum equilibrium yield condition. For $f_{eq} = 0.005$, the equilibrium values are: $N_{eq} = 0.16$, $E_{eq} = 20.3$, $W_{eq} = W_{h_{eq}} = 0.8$. Following the procedures outlined in the Appendix (Equations (A-2) through (A-4)), Equations (9), (10), and (11) were linearized around this equilibrium to obtain:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0.03 & 0 & -0.56 \\ -5.73 & -0.12 & -0.81 \\ 0.14 & 0.02 & -2.02 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u \tag{12}$$

where: $x_1 = N - N_{eq}$

$x_2 = E - E_{eq}$

$x_3 = W - W_{eq}$

$u = f - f_{eq}.$

The following performance index J describes our desire to keep the system near the desired equilibrium (Appendix, Equation (A-6)):

$$J = \int_0^\infty (Q_{11}x_1^2 + Q_{22}x_2^2 + Q_{33}x_3^2 + Ru^2)\, dt. \tag{13}$$

Here the weighting matrix $[Q]$ becomes:

$$[Q] = \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}$$

and the matrix $[R]$ becomes a scalar R. A substantial difference between the single-variable and multivariable cases is that in the latter case we can no longer easily determine the feedback gains by specifying the desired values of the time constants. Instead, the gains are calculated by specifying the components of the weighting matrices. A common procedure for doing this is to choose the components by the rule:

$$Q_{11} = \frac{1}{x_{1m}^2} \qquad R = \frac{1}{u_m^2}$$

where x_{1m} is the maximum desired magnitude of the deviation x_1 of the population N, and u_m is the maximum desired magnitude of the deviation u of the fishing rate f. The components Q_{22} and Q_{33} are chosen in a similar manner. Here we assume the maxima are specified to be:

$$\begin{aligned} x_{1m} &= 5\% \text{ deviation from } N_{eq} = 0.008 \\ x_{3m} &= 1\% \text{ deviation from } W_{eq} = 0.008 \\ u_m &= 50\% \text{ deviation from } f_{eq} = 0.0025. \end{aligned}$$

Thus:

$$\frac{1}{x_{1m}^2} = \frac{1}{x_{3m}^2} = 1.6 \times 10^4 = 0.1\,\frac{1}{u_m^2}.$$

Assuming that the variation x_2 in the food density is not of direct interest, we set $Q_{22} = 0$. Since J from Equation (13) depends only on the relative magnitudes of the weighting factors, we can choose these factors to be:

$$\begin{aligned} R &= 1 \\ Q_{11} &= Q_{33} = 0.1 \\ Q_{22} &= 0. \end{aligned}$$

For this three-variable model, the symmetric Riccati matrix $[P]$ has nine elements, three of which are redundant. Computer solution of the six coupled differential equations resulting from Equation (A-9) and use of Equations (A-7) and (A-8) yield the following feedback control function:

$$u = -[K]\underline{x} = 0.149x_1 + 0.00027x_2 + 0.026x_3. \quad (14)$$

Use of Equation (14) and the definitions of x_1, x_2, x_3, and u gives the optimal fishing effort as a feedback function of the system variables:

$$f = f_{eq} + 0.149\,(N - N_{eq}) + 0.00027\,(E - E_{eq}) + 0.026\,(W - W_{eq}).$$

Substitution of the equilibrium values gives:

$$f = -0.045 + 0.149N + 0.00027E + 0.026W. \quad (15)$$

Substitution of u from Equation (14) into Equation (12) gives the set of linearized equations describing the behavior of the model under feedback control. The matrix $[A]$ to be used in Equation (A-5) becomes

$$[A] = \begin{bmatrix} -0.119 & -0.00027 & 0.534 \\ -5.73 & -0.12 & -0.81 \\ 0.14 & 0.02 & -2.02 \end{bmatrix}$$

The roots of Equation (A-5) are: $s = -2.09, -0.096 \pm 0.17j$, where $j = \sqrt{-1}$. The dominant time constant is the negative reciprocal of the least negative real part, and here is equal to $1/0.096 = 10.4$ time units. In a similar way the dominant time constant for the system without feedback is 45.5 time units. Thus the feedback control given by Equation (15) reduces the effects of disturbances in one-fourth the time. These linearized results have been verified by simulation of the original nonlinear model. Other simulations are discussed by Palm (1975).

Before concluding this example, we note from Equation (15) that f is a function of all three variables. This is due to the coupling between the three equations. Also, although the choice of the weighting factors is somewhat arbitrary, this should not obscure the fact that the Linear-Quadratic Optimal Control Problem provides a systematic method for determining the feedback gain matrix $[K]$. A systematic approach is needed because the number of components of $[K]$ becomes so large for multivariable problems that a trial-and-error approach is prohibitive. As long as $[Q]$ and $[R]$ are chosen to be positive-definite, the resulting $[K]$ will stabilize the system. Various choices of $[Q]$ and $[R]$ merely affect the time constants and form of response (oscillatory vs. nonoscillatory return to equilibrium). This is the main advantage of this technique.

With this model the effects of mesh size regulation can be studied by using W_h as an additional control variable. Also, the food supply rate σ is another possible control variable if the model is used to analyze fish farming. The linear-quadratic control technique could be used in both cases.

CONCLUSION

In this introductory paper we have presented only the deterministic case of the Linear-Quadratic Optimal Control Problem. In order to set limits on fishing effort which are functions of system variables such as population density or mean organism weight, it is necessary to measure these variables. Any measurement process is stochastic or noisy, and it is necessary to compensate for this in the design of a feedback regulation scheme. In many engineering applications this has been successfully accomplished by the use of the Kalman-Bucy filter (Athans 1971). In addition, it may be impossible even to measure some variables. This problem of incomplete information has been frequently solved by the use of the Observer Theory (Kwakernaak and Sivan 1972).

Also there will be uncertainties in the determination of the model constants. In fact the "constants" may not be constants at all, but merely the representation of several effects lumped together. Thus there is also error in the model structure, since the model constants are actually variables dependent upon a variety of effects. For the Schaefer model these effects would be interspecies interactions, age structure, availability and vulnerability of the age groups, and physical environment influences on the biological processes. Such difficulties are amenable to solution by adding a "noise" term to the model equations and by modifying the linear-quadratic techniques to accommodate these stochastic effects (Athans 1971). It should also be pointed out that compensation for modeling errors is one of the purposes of feedback control.

The change in model parameters with time can be compensated for by regularly recomputing the feedback gains as more data becomes available. Finally, while no pretense is made of being able to predict exact time paths, the methods described in this paper should prove useful in providing management guidelines. The effects of stochastic processes and uncertainties can be handled in a manageable way by computer simulation, and prediction of the future course of the managed fishery, in an average sense, can be made with appropriate error bands placed on the predictions.

ACKNOWLEDGMENT

I thank Saul Saila for bringing Goh's results to my attention, and the referees for their comments on the bang-bang control approach and the effects of uncertainties in model parameters.

LITERATURE CITED

ATHANS, M.
1971. The role and use of the stochastic linear-quadratic-gaussian problem in control system design. Inst. Electr. Electron. Eng., Trans. Autom. Control. AC-16:529-551.

BEVERTON, R. J. H., AND S. J. HOLT.
1957. On the dynamics of exploited fish populations. Fish Invest. Minist. Agric. Fish. Food (G. B.), Ser. II, 19, 533 p.

CARROTHERS, W. A.
1941. The British Columbia fisheries. Univ. Toronto Press, Toronto, 136 p.

CLARK, C., G. EDWARDS, AND M. FRIEDLANDER.
1973. Beverton-Holt model of a commercial fishery: Optimal dynamics. J. Fish. Res. Board Can. 30:1629-1640.

CLARK, C. W.
1973. The economics of overexploitation. Science (Wash., D.C.) 181:630-634.

FOX, W. W., JR.
1970. An exponential surplus-yield model for optimizing exploited fish populations. Trans. Am. Fish. Soc. 99:80-88.

GOH, B. S.
1969. Optimal control of the fish resource. Malayan Sci. 5:65-70.
1973. Optimal control of renewable resources and populations. (Summary of paper presented at the 6th Hawaii International Conference on System Sciences) 5 p.

HO, Y. C., AND A. E. BRYSON.
1969. Applied optimal control: Optimization, estimation and control. Blaisdell, Waltham, Mass., 481 p.

KWAKERNAAK, H., AND R. SIVAN.
1972. Linear optimal control systems. Wiley - Interscience, N.Y., 575 p.

LARKIN, P. A.
1963. Interspecific competition and exploitation. J. Fish. Res. Board Can. 20:647-678.
1966. Exploitation in a type of predator-prey relationship. J. Fish. Res. Board Can. 23:349-356.

PALM, W. J.
1975. An application of control theory to population dynamics. *In* E. O. Roxin, P. T. Liu, and R. L. Sternberg (editors), Differential games and control theory, p. 59-70. Marcel-Dekker, N.Y.

PELLA, J. J., AND P. K. TOMLINSON.
1969. A generalized stock production model. Inter-Am. Trop. Tuna Comm., Bull. 13:419-496.

RICKER, W. E.
1958. Handbook of computations for biological statistics of fish populations. Fish. Res. Board Can., Bull. 119, 300 p.

SAILA, S. B.
In press. Some applications of optimal control theory to fisheries management. Trans. Am. Fish. Soc.

SCHAEFER, M. B.
1954. Some aspects of the dynamics of populations important to the management of commercial marine fisheries. Inter-Am. Trop. Tuna Comm., Bull. 1:25-56.
1968. Methods of estimating effects of fishing on fish populations. Trans. Am. Fish. Soc. 97:231-241.

TIMIN, M. E., AND B. D. COLLIER.
1971. A model incorporating energy utilization for the dynamics of single species populations. Theor. Popul. Biol. 2:237-251.

APPENDIX

The Linear-Quadratic Optimal Control Problem and its solution are now outlined. For a thorough discussion, see Ho and Bryson (1969) or Kwakernaak and Sivan (1972). By use of state variable notation, any set of time-invariant ordinary differential equations can be put into the following form:

$$\frac{d\underline{y}}{dt} = \underline{g}\,(\underline{y},\underline{f}) \qquad \text{(A-1)}$$

where $\underline{g}$ is a general n-dimensional vector function, $\underline{y}$ is the n-dimensional state vector for the model, and $\underline{f}$ is the m-dimensional input or forcing function vector. If this system has an equilibrium ($\underline{y}_{eq}, \underline{f}_{eq}$, the following set of algebrais equations must be satisfied:

$$\underline{0} = \underline{g}\,(\underline{y}_{eq}, \underline{f}_{eq}).$$

The values of $\underline{y}_{eq}$ and $\underline{f}_{eq}$ depend on the system's parameters, and static optimazation methods such as calculus or linear programming can be applied to find the optimal $\underline{y}_{eq}$, $\underline{f}_{eq}$ and system parameters according to some criterion. This was done in the first example to determine the condition of maximum equilibrium yield. Since any real system is subjected to varying conditions and disturbances, it will be continually displaced from equilibrium. Thus for unstable systems or stable systems with large time constants, a static method of analysis is not sufficient. In such a case the next step is to apply a dynamic optimization method, such as the method presented here, to devise a control scheme which ensures that the system will return to equilibrium with a satisfactory time constant. Thus static and dynamic methods should not be viewed as alternative approaches to optimization, but rather as mutually complementary methods.

After the equilibrium is determined, Equation (A-1) is linearized by expanding the function $\underline{g}$ in a Taylor series in $\underline{y}$ and $\underline{f}$, and keeping only the first-order terms. This gives the linearized model:

$$\frac{d\underline{x}}{dt} = [A]\,\underline{x} + [B]\,\underline{u} \qquad \text{(A-2)}$$

where: $\underline{x} = \underline{y} - \underline{y}_{eq}$

$\underline{u} = \underline{f} - \underline{f}_{eq}$

$$[A] = \left[\left(\frac{\delta \underline{g}}{\delta \underline{y}}\right)_{eq}\right] \qquad \text{(A-3)}$$

$$[B] = \left[\left(\frac{\delta \underline{g}}{\delta \underline{f}}\right)_{eq}\right] \qquad \text{(A-4)}$$

in which the subscript eq indicates that the partial derivatives of $\underline{g}$ are evaluated at the equilibrium. The stability of the equilibrium can be determined from the roots of the determinant equation:

$$\Big| s\,[I] - [A] \Big| = 0 \qquad \text{(A-5)}$$

where $[I]$ is the $(n \times n)$ identity matrix. The equilibrium is stable if and only if all of the roots s have negative real parts.

By finding the function $\underline{u}$ which minimizes the following quadratic performance index, $\underline{x}$ and $\underline{u}$ are kept near zero and thus the system is kept near equilibrium.

$$J = \int_0^\infty (\underline{x}^T [Q]\,\underline{x} + \underline{u}^T [R]\,\underline{u})\,dt. \qquad \text{(A-6)}$$

The feedback control function which minimizes J has been shown to be:

$$\underline{u} = -[K]\,\underline{x}. \qquad \text{(A-7)}$$

The feedback "gain" matrix $[K]$ is calculated from:

$$[K] = [R]^{-1} [B]^T [P] \qquad \text{(A-8)}$$

where the Riccati matrix $[P]$, an $(n \times n)$ symmetric matrix, is the steady-state solution of the Riccati matrix differential equation:

$$\frac{d[P]}{dt} = [Q] + [A]^T[P] + [P][A] - [P][B][R]^{-1}[B]^T[P] \qquad \text{(A-9)}$$

with the initial condition:

$$[P(0)] = [0].$$

The matrix $[P]$ is usually found by numerically solving the Riccati equation until all the components of the solution $[P]$ become constant. This will always occur.