A MODEL FOR OPTIMAL SALMON MANAGEMENT[1]

DOUGLAS E. BOOTH[2]

ABSTRACT

Considerable attention has been given in the literature recently to continuous time dynamic maximizing models for fisheries in general, but the time discreteness and interdependency problems encountered in the case of most salmon fisheries have been largely ignored. Hence, a discrete time profit maximizing model for a salmon fishery is developed in this paper, and it is shown that a correct salmon management policy takes the form of an investment decision with respect to the level of escapement and that a management policy of maximum sustained yield may be incorrect from an economic standpoint. It is hoped that continued research including construction of a working model will provide some indication of the difference between the magnitude of spawner stocks selected on the basis of maximum sustained yield and stocks selected on the basis of economic optimality.

Continuous time dynamic maximizing models have been developed in the literature recently to handle the problem of optimally managing a fishery resource (Brown, 1969[3]; Quirk and Smith, 1970[4]). The continuous time approach to analyzing management policy for a salmon fishery tends to be unrealistic since the reproductive process for salmon is periodic, and for certain species reproduction involves rather complex time interdependencies. In the simplest case salmon spawned in a given time period will return to their spawning ground in some future time period, while in more complex cases salmon spawned in a given time period will return to their spawning grounds in several different runs over a number of time periods; also, the level of spawning activity in one time period may affect the fertility of the spawning grounds in future time periods. Such discreteness and time interdependencies cannot be adequately characterized in a continuous time mathematical model.

Hence, the purpose of this paper is to develop a discrete time maximizing model based on currently accepted views of biological spawner-return relationships for salmon; the model is developed with the biological properties of the Bristol Bay fishery foremost in mind (Mathews, 1967). It is shown that a correct fishery management policy takes the form of an investment decision with respect to the level of escapement and that a management policy of maximum sustained yield may be incorrect from an economic standpoint. In essence the fishery manager must decide whether to invest in spawners which yield a return of additional fish at future points in time, or to catch and sell potential spawners today.

In the first section of the paper, the notation and assumptions of the analysis are presented, and in the second section, a simple first-order difference equation model of a salmon fishery is developed and discussed. In the third section, the model is extended to account for the fact that salmon spawned in time period t will return to the spawning grounds in time periods $t+4$, $t+5$, and $t+6$, and also to account for the possibility that fish spawned in time period t will deplete the spawning grounds of food to such an extent that the number of fish the spawning grounds can support in time period $t+1$ will be reduced. Desirable refinements and applications of the model

[1] Contribution No. 361, College of Fisheries, University of Washington.

[2] Fisheries Research Institute, 260 Fisheries Center, University of Washington, Seattle, WA 98195.

[3] Brown, G. W., Jr., 1969. An optimal program for managing common property resource with congestion externalities. Univ. Washington, Seattle. [Mimeograph.]

[4] For a static linear-programming model useful for analyzing fishery management problems see Rothschild and Balsiger, 1971.

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are discussed in the concluding section. In the Appendix the continuous time analog to the simple model is presented.

NOTATION AND ASSUMPTIONS

The notation to be used is as follows:

R_t = the size of a run of salmon into a given river in time period t;
S_t = the number of spawners allowed to escape up the river in time period t;
x_t = the catch of salmon from a given river in time period t;
E_t = the amount of effort used to catch x_t salmon in time period t;
P_E = the price per unit of effort;
P_x = the price per unit of fish;
r = the appropriate discount rate;
T = the total number of years.

The assumptions of the analysis are as follows:

(i) The industry catch for a given river, x_t, is a linear homogeneous function of the amount of effort employed E_t and the size of the run R_t:

$$\begin{aligned} x_t &= F(E_t, R_t) \\ &= R_t\ F(E_t/R_t,\ 1) \\ &= R_t\ f(k_t);\ \text{ where } k_t = E_t/R_t. \end{aligned}$$ [5]

(ii) The biological spawner-return relationship is of the form developed by Ricker (Mathews, 1967):

$$R_t = a\ S_{t-1}\ e^{(-bS_{t-1})}.$$

A graph of this function for a simple first order model appears in Figure 1. If a policy of maximum sustained yield is followed, the escapement in year t—1, $S^0{}_{t-1}$, occurs where $R_t - S_{t-1}$ is a maximum, or where

$$\frac{d\ (S_{t-1}\ a\ e^{(-bS_{t-1})} - S_{t-1})}{d\ S_{t-1}} =$$

$$a\ e^{(-bS_{t-1})}(1 - b\ S_{t-1}) - 1 = 0.$$

[5] The first derivative of f will be denoted f'.

The escapement in year t is:

$$S_t = R_t - R_t\ f\ (k_t).$$

(iii) The appropriate objective function to maximize is assumed to be

$$\sum_{t=0}^{T-1} \frac{1}{(1+r)^t}\Big[R_t\ f(k_t)\ P_x - R_t k_t P_E \Big] + \frac{1}{(1+r)^T}\ G(R_T)$$

where the expression on the left is the present worth of industry profits over T—1 years, and the second expression is the present worth of a value function for the terminal stock of fish.

(iv) The price of fish P_x and the price of effort P_E are assumed to remain constant for all time periods.

For some readers the purpose of making these assumptions may at this point appear unclear. Hopefully, the comments to follow will clarify any ambiguities.

In assumption (i) a linear homogeneous aggregate production function is selected for its convenient mathematical properties, and because it has an important economic property, constant returns to scale. In most industry aggregate production function studies, the assumption of constant returns has been found to be reasonably realistic. However, in the case of the salmon industry, this hypothesis has yet to be tested.

The spawner-return relationship specified in assumption (ii) has the usual properties of fishery recruitment functions. It is clear from the graph that spawner stocks to the right of the point $\bar{S}_{t-1}$, where R_t is a maximum as a function of S_{t-1}, are irrelevant from a policy standpoint, since for any feasible run size R_t there corresponds a spawner stock S_{t-1} with $S_{t-1} \leqslant \bar{S}_{t-1}$. Note that no species interaction is implicitly assumed.

The assumption that the present worth of industry profits is the appropriate objective function to maximize is an assumption commonly made in economic analysis. Other types of ob-

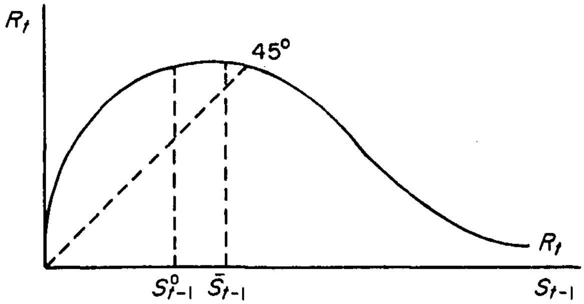

FIGURE 1.—The Ricker spawner-return relationship for salmon.

jective functions are mathematically feasible, but it is not obvious what they might be without going beyond the scope of this paper and undertaking a political analysis of salmon fishery management. Specifying a terminal value function is a mathematical necessity and will be discussed later.

Assumption (iv) implies, among other things, that the catch x_t from the salmon stock being analyzed is not large enough to influence the total industry price structure for salmon and that the factor supply market for effort, E_t, is competitive.

The problem of gear congestion on the fishing ground is adequately dealt with elsewhere and is avoided here by assuming that the fishery management authority undertakes appropriate policies to insure that only efficient levels of effort are employed (see footnote 3; Quirk and Smith, 1970). It is also assumed that salmon are not caught on the high seas but are harvested as they return to the river to spawn.

A SIMPLE MODEL

In this section the simplest type of spawner-return relationship is examined where salmon spawned in time period t—1 return to their spawning ground in time period t. Given the assumptions of the above section, the problem is to maximize

$$\sum_{0}^{T-1} \frac{1}{(1+r)^t}\Big(R_t\, f(k_t)\, P_x - R_t k_t P_E\Big) + \frac{1}{(1+r)^T}\, G(R_T),$$

subject to:

$$S_t = R_t - R_t\, f(k_t),\, S_t \geqslant 0,\, k_t \geqslant 0,\, R(0) = R_0,$$

and

$$R_t = a\, S_{t-1}\, e^{(-bS_{t-1})}.$$

The appropriate Lagrangian for the maximization problem is:

$$L\ (k, S, \lambda) = \sum_{0}^{T-1} \frac{1}{(1+r)^t}\Big(R_t\ (P_x\, f(k_t) - P_E\, k_t)\Big) + \frac{1}{(1+r)^T}\, G(R_T) + \sum_{0}^{T-1} \frac{1}{(1+r)^t}\, \lambda_t[R_t - R_t\, f(k_t) - S_t], \quad [1]$$

where $k = (k_0, \ldots, k_{T-1})$, $S = (S_0, \ldots, S_{T-1})$, and $\lambda = (\lambda_0, \ldots, \lambda_{T-1})$, and R_0 is given.

The appropriate Kuhn-Tucker necessary conditions for a maximum of $L\ (k, S, \lambda)$ are as follows:

$$\partial L/\partial k_t \leqslant 0;\ k_t\ \partial L/\partial k_t = 0, \quad t = 0, \ldots, T-1; \quad [2]$$

$$\partial L/\partial S_t \leqslant 0;\ S_t\ \partial L/\partial S_t = 0, \quad t = 0, \ldots, T-1; \quad [3]$$

$$\partial L/\partial \lambda_t = 0, \quad t = 0, \ldots, T-1. \quad [4]$$

It is assumed that the solution to the maximizing problem is interior; i.e., $k_t > 0, S_t > 0, \lambda_t > 0$. Then [2] and [3] are satisfied if [5], [6], and [7] are true:

$$(P_x - \lambda_t)\, f'(k_t) = P_E, \quad t = 0, \ldots, T-1; \qquad [5]$$

$$\lambda_{t-1} = \frac{1}{(1+r)} \left(P_x\, f(k_t) - P_E\, k_t\right) dR_t/dS_{t-1} + \frac{1}{(1+r)} \lambda_t \,[1 - f(k_t)]\, dR_t/dS_{t-1},$$

$$t = 1, \ldots, T{-}1; \qquad [6]$$

$$\lambda_{T-1} = \frac{1}{(1+r)} \frac{dG}{dR_T} \frac{dR_T}{dS_{T-1}}, \qquad [7]$$

and [4] is satisfied if [8] is true.

$$S_t = R_t - R_t\, f(k_t), \quad t = 0, \ldots, T{-}1. \qquad [8]$$

For $t = 1$ equation [6] can be written more simply as

$$\lambda_0 = \frac{1}{1+r}[P_x\, f(k_1) - P_E\, k_1]\, \frac{dR_1}{dS_0} + \frac{1}{1+r}\, \lambda_1 \frac{\partial S_1}{\partial S_0}\,. \qquad [6']$$

By substituting for λ_1 in [6′], an expression of λ_0 results with λ_2 in it, and by substituting for λ_2 an expression of λ_0 results with λ_3 in it, and so on until an expression of λ_0 results with λ_{T-1} in it. Equation [7] can then be substituted for λ_{T-1}.[6] The resulting expression is:

$$\lambda_0 = \sum_{t=1}^{T-1} \frac{1}{(1+r)^t} \left[P_x\, f(k_t) - P_E\, k_t \right] \frac{\partial R_t}{\partial S_0} + \frac{1}{(1+r)^T} \frac{dG}{dR_T} \frac{\partial R_T}{\partial S_0}\,. \qquad [9]$$

Equation [5] can be rewritten for $t = 1$ as

$$\lambda_0 = P_x - P_E/f'(k_0). \qquad [5']$$

Equation [5′] suggests that λ_0 can be interpreted as the marginal profitability of catching an additional fish in period 0, while equation [9] suggests that λ_0 can be interpreted as the present value of the marginal profitability of adding an additional spawner to the escapement level in period 0; i.e., the level of escapement should be selected in time period 0 such that the incremental profitability of an additional fish caught today is just equal to the present value of the profitability of the future return resulting from an incremental spawner. In order to attain the desired level of escapement it is necessary to select an appropriate level of k_0, since escapement is equal to the run size in period 0 minus the catch. Note that this analysis can be applied to any time period t, not just to $t = 0$.

In order for the analysis to be valid in the general form presented here, it is necessary to prove the existence of values for $k = (k_0, \ldots, k_{T-1})$, $S = (S_0, \ldots, S_{T-1})$, and $\lambda = (\lambda_0, \ldots, \lambda_{T-1})$, which satisfy equations [5], [6], [7], and [8] given $R_0 = R(0)$. This is not an easy task if T is finite, so it is assumed that $T \to \infty$.[7] In this situation an equilibrium, or steady state solution is possible, where

$$k^* = k_0 = k_1 = \ldots, = k_T$$

$$S^* = S_0 = S_1 = \ldots, = S_T$$

$$\lambda^* = \lambda_0 = \lambda_1 = \ldots, = \lambda_T.$$

If T is finite this kind of a solution makes no sense. If $T \to \infty$ and an equilibrium solution exists, then equation [9] can be rewritten as

[6] This procedure is discussed in Burt and Cummings (1970).

[7] For a discussion of the mathematical problems involved in the case of an infinite time horizon (Burt and Cummings, 1970).

$$\lambda^* = \sum_{t=1}^{\infty} \frac{1}{(1+r)^t} [P_x f(k^*) - P_E k^*] \frac{\partial R_t^*}{\partial S_0} + \lim_{T \to \infty} \frac{1}{(1+r)^T} \frac{dG}{dR_T} \frac{\partial R_T^*}{\partial S_0}.^{8} \qquad [9']$$

Both expressions to the right of the equal sign in equation [9′] must converge for existence. This requires that

$$\frac{dR^*}{dS} [1 - f(k^*)] < 1 + r. \qquad [10]$$

Then equation [9′] can be rewritten as

$$\lambda^* = \frac{1}{1+r} [P_x f(k^*) - P_E k^*] \frac{dR^*}{dS} \cdot \frac{1}{1 - \frac{[1 - f(k^*)]}{1+r} \frac{dR^*}{dS}} \qquad [9'']$$

since $\lim_{T \to \infty} \frac{1}{(1+r)^T} \frac{dG}{dR_T} \frac{\partial R_T^*}{\partial S_0}$ converges to zero for $\frac{dG}{dR_T}$ bounded. Also [5] and [8] can be rewritten as

$$(P_x - \lambda^*) f'(k^*) = P_E, \qquad [5']$$

$$S^* = R^* - R^* f(k^*). \qquad [8']$$

Equations [10] and [9″] together suggest that the productivity of the spawner stock in producing additional spawners must be less than the social rate of discount, r, in order for λ^*, the profitability of an incremental spawner, to be positive.

A steady state solution exists if the determinant of the Jacobian matrix for [5′], [8′], and [9″], is non-zero. It can be proven that the Jacobian determinant is negative if it is assumed that the expression to the left of the inequality sign in [10] is less than or equal to 1. Unfortunately, there is no reason to assume this, even though in the context of a specific model, with functions and parameters assigned, one might expect that it is true.[9]

Note that the equilibrium solution requires that $R_0 = S^*/(1 - f(k^*))$. Clearly there is no reason to anticipate that the initial run size will be at the desired level for an equilibrium solution, and to reach the desired level may be costly. Hence, the analysis ignores the question of optimal policy for reaching the equilibrium solution. Note also that there is no reason to anticipate that S^*, the equilibrium solution, will correspond to maximum sustained yield. These issues are discussed more extensively in the appendix in the context of a continuous time model.

MORE COMPLEX MODELS

A simple model of the type presented in the previous section is not completely realistic for certain species of salmon where fish spawned in time period t will return to their spawning grounds in time periods $t+4$, $t+5$, and $t+6$. Assuming that the percentages of the total return to the river in time periods $t+4$, $t+5$, and $t+6$ from spawning efforts in time period t are constants, a_4, a_5, and a_6, the spawner-return relationship becomes:

$$R_t = S_{t-4}\, a_4\, a\, e^{(bS_{t-4})} + S_{t-5}\, a_5\, a\, e^{(-bS_{t-5})} + S_{t-6}\, a_6\, a\, e^{(-bS_{t-6})}, \qquad [11]$$

where $a_4 + a_5 + a_6 = 1$.

[8] In a steady state solution,

$$\frac{\partial R_t}{\partial S_0} = \frac{dR^*}{dS} \left([1 - f(k^*)] \frac{dR^*}{dS} \right)^{t-1}$$

where * denotes evaluation at the equilibrium solution. Note also that

$$\frac{\partial R_t}{\partial S_1} = \frac{dR_t}{dS_{t-1}} \prod_{\tau=1}^{t-1} [1 - f(k_\tau)] \frac{dR_\tau}{dS_{\tau-1}} = \frac{dR_t}{dS_{t-1}} \prod_{\tau=1}^{t-1} \frac{\partial S_\tau}{\partial S_{\tau-1}} \text{ in the general case.}$$

[9] See the appendix for a proof that the Jacobian determinant is negative for the given assumption.

The Lagrangian for the maximizing problem now becomes:

$$L = \sum_{t=0}^{T-6} \frac{1}{(1+r)^t} \left(R_t \left[P_x f(k_t) - P_E k_t \right) + \lambda_t \left[R_t - R_t f(k_t) - S_t \right] \right)$$

$$+ \sum_{t=T-5}^{T} \frac{1}{(1+r)^t} G(R_t), \qquad [12]$$

given $R(0) = R_0$, $R(1) = R_1$, $R(2) = R_2$, $R(3) = R_3$, $R(4) = R_4$, and $R(5) = R_5$. Note that in this model R_0, R_1, R_2, R_3, R_4, and R_5 are the given initial run sizes and that the terminal value condition must be modified to account for the more complex spawner-return relationship now being used. Also, it is implicitly assumed that run sizes in the last five periods have no direct effect on one another, but are determined by spawner stocks in previous periods. Once $T \rightarrow \infty$ is allowed this assumption becomes unimportant.

The necessary conditions for a maximum of L with $k_t > 0$, $S_t > 0$, $\lambda_t > 0$ are satisfied if:

$$(P_x - \lambda_t) f'(k_t) = P_E, \quad t = 0, \ldots, T-6; \qquad [13]$$

$$\lambda_{t-4} = \frac{1}{(1+r)^4} \left([P_x f(k_t) - P_E k_t] + \lambda_t [1 - f(k_t)] \right) \frac{\partial R_t}{\partial S_{t-4}}$$

$$+ \frac{1}{(1+r)^5} \left([P_x f(k_{t+1}) - P_E k_{t+1}] + \lambda_{t-1} [1 - f(k_{t+1})] \right) \frac{\partial R_{t+1}}{\partial S_{t-4}}$$

$$+ \frac{1}{(1+r)^6} \left([P_x f(k_{t+2}) - P_E k_{t+2}] + \lambda_{t-2} [1 - f(k_{t+2})] \right) \frac{\partial R_{t+2}}{\partial S_{t-4}},$$

$$t = 4, \ldots, T-8; \qquad [14]$$

$$\lambda_{T-11} = \frac{1}{(1+r)^4} \left([P_x f(k_{T-7}) - P_E k_{T-7}] + \lambda_{T-7} [1 - f(k_{T-7})] \right) \frac{\partial R_{T-7}}{\partial S_{T-11}}$$

$$+ \frac{1}{(1+r)^5} \left([P_x f(k_{T-6}) - P_E k_{T-6}] + \lambda_{T-6} [1 - f(k_{T-6})] \right) \frac{\partial R_{T-6}}{\partial R_{T-11}}$$

$$+ \frac{1}{(1+r)^6} \frac{dG}{dR_{T-5}} \frac{\partial R_{T-5}}{\partial S_{T-11}}; \qquad [15]$$

$$\lambda_{T-10} = \frac{1}{(1+r)^4} \left([P_x f(k_{T-6}) - P_E k_{T-6}] + \lambda_{T-6} [1 - f(k_{T-6})] \right) \frac{\partial R_{T-6}}{\partial S_{T-10}}$$

$$+ \frac{1}{(1+r)^5} \frac{dG}{dR_{T-5}} \frac{\partial R_{T-5}}{\partial S_{T-10}} + \frac{1}{(1+r)^6} \frac{dG}{dR_{T-4}} \frac{\partial R_{T-4}}{\partial S_{T-10}}; \qquad [16]$$

$$\lambda_{T-9} = \frac{1}{(1+r)^4} \frac{dG}{dR_{T-5}} \frac{\partial R_{T-5}}{\partial S_{T-9}} + \frac{1}{(1+r)^5} \frac{dG}{dR_{T-4}} \frac{\partial R_{T-4}}{\partial S_{T-9}} + \frac{1}{(1+r)^6} \frac{dG}{dR_{T-3}} \frac{\partial R_{T-3}}{\partial S_{T-9}}; \qquad [17]$$

$$\lambda_{T-8} = \frac{1}{(1+r)^4} \frac{dG}{dR_{T-4}} \frac{\partial R_{T-4}}{\partial S_{T-8}} + \frac{1}{(1+r)^5} \frac{dG}{dR_{T-3}} \frac{\partial R_{T-3}}{\partial S_{T-8}} + \frac{1}{(1+r)^6} \frac{dG}{dR_{T-2}} \frac{\partial R_{T-2}}{\partial S_{T-8}}; \qquad [18]$$

$$\lambda_{T-7} = \frac{1}{(1+r)^4} \frac{dG}{dR_{T-3}} \frac{\partial R_{T-3}}{\partial S_{T-7}} + \frac{1}{(1+r)^5} \frac{dG}{dR_{T-2}} \frac{\partial R_{T-2}}{\partial S_{T-7}}$$

$$+\frac{1}{(1+r)^6}\frac{dG}{dR_{T-1}}\frac{\partial R_{T-1}}{\partial S_{T-7}}; \quad [19]$$

$$\lambda_{T-6} = \frac{1}{(1+r)^4}\frac{dG}{dR_{T-2}}\frac{\partial R_{T-2}}{\partial S_{T-6}} + \frac{1}{(1+r)^5}\frac{dG}{dR_{T-1}}\frac{\partial R_{T-1}}{\partial S_{T-6}} + \frac{1}{(1+r)^6}\frac{dG}{dR_T}\frac{\partial R_T}{\partial S_{T-6}}; \quad [20]$$

$$S_t = R_t[1-f(k_t)], t=0,\ldots,T-6. \quad [21]$$

For a steady state (k^*, S^*, λ^*) these necessary conditions reduce to the following:

$$(P_x - \lambda^*)\, f'(k^*) = P_E, \quad [22]$$

$$\lambda^* = [P_x f(k^*) - P_E\, k^*]\frac{dR^*}{dS} \times \frac{D}{1-D[1-f(k^*)]\frac{dR^*}{dS}} \quad [23]$$

$$S^* = R^*[1-f(k^*)], \quad [24]$$

where $D = \frac{a_4}{(1+r)^4} + \frac{a_5}{(1+r)^5} + \frac{a_6}{(1+r)^6}$.

In order for λ^* to be positive, $[1 - f(k^*)]\frac{dR^*}{dS}$ must be less than $1/D$. The economic interpretations of [22], [23], and [24] are the same as the corresponding interpretations of [5′], [9″], and [8′] in the second section.

Some biologists believe that for certain species and spawning grounds spawned salmon in time period t will deplete the spawning grounds of food to such a degree that food sources will not be replenished sufficiently in time period $t+1$ to support an equally large number of spawned fish. The nature of this phenomenon has not yet been very well specified, but one possible expression of it is the following spawner-return relationship where feeding interaction between years is accounted for by modification of the power term for e:

$$R_t = S_{t-4}\, a_4 a e^{(-b_1 S_{t-4} - b_2 S_{t-5})} + S_{t-5}\, a_5 a e^{(-b_1 S_{t-5} - b_2 S_{t-6})} + S_{t-6}\, a_6 a e^{(-b_1 S_{t-6} - b_2 S_{t-7})}. \quad [25]$$

The necessary conditions in a steady state for the model using this spawner-return relationship are the same as [22], [23], and [24], except that $\frac{dR^*}{dS}$ must be replaced by $\frac{dR^*}{dS} = ae^{-(b_1 + b_2)S^*}\left(1 - b_1 S^* - \frac{b_2}{1+r}S^*\right)$.

One thing should be noted about the two models presented in this section. In order to attain the steady state solution, it is necessary to set the run size equal to its equilibrium solution level for the first six periods in the first model and the first seven periods in the second model. Hence, the problem of attaining the equilibrium solution is of greater magnitude here than in the simple model, where it was necessary to set the run size equal to its equilibrium solution level only in the initial period. This problem can, perhaps, be more adequately dealt with in the framework of a specific model and, in any case, requires further research.

NEEDED REFINEMENTS

Thus far, the analysis considers only necessary conditions for the existence of a maximizing solution. Sufficient conditions for existence are satisfied if the Lagrangian is concave in all variables. Unfortunately, concavity neither can be proved or disproved for the models examined in this paper. Again, the proof may be possible in the context of a more specific model, where all parameters and functions are assigned.

A practical application of the model presented here would involve significantly difficult estimation problems. Some work has already been done on estimating spawner-return functions, but the results in most cases have not been too promising (Mathews, 1967). The spawner-return relationships seem to possess a high degree of variability, which suggests that a stochastic specification of the problem may be more realistic than our deterministic approach. Estimating a catch function may also be troublesome since it would require a careful specification of effort and of its price.

Despite the difficulties, construction of a working model would be worthwhile because it would provide insight into some of the unresolved mathematical problems mentioned above, and more importantly it would provide further information on the significance of the difference between the magnitude of spawner stocks selected on the basis of maximum sustained yield and stocks selected on the basis of economic optimality as defined in this paper.

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APPENDIX

A continuous time analog to the simple model will now be discussed briefly. The problem is to maximize.

$$\int_0^\infty R_t[P_x f(k_t) - P_E k_t] e^{-\rho t} dt \qquad [A\text{-}1]$$

subject to

$$\dot{S}_t = R_t[1 - f(k_t)] - S_t, \qquad [A\text{-}2]$$

where $\dot{S}_t = \frac{dS_t}{dt}$, and ρ is the continuous time discount rate. The appropriate Hamiltonian for the maximum problem is

$$H = e^{-\rho t} \Big(R_t[P_x f(k_t) - P_E k_t] + \lambda_t [R_t[1\text{-}f(k_t)] - S_t - \dot{S}_t] \Big). \qquad [A\text{-}3]$$

Assuming the existence of an interior solution, the necessary conditions for a maximum are, along with [A-2],

$$\frac{\partial H}{\partial k_t} = 0, \qquad [A\text{-}4]$$

$$\frac{d}{dt} \frac{\partial H}{\partial \dot{S}_t} = \frac{\partial H}{\partial S_t}. \qquad [A\text{-}5]$$

[A-4] and [A-5] are satisfied if

$$(P_x - \lambda_t) f' = P_E, \qquad [A\text{-}4']$$

$$\dot{\lambda}_t = \rho\lambda_t - \Big([P_x f(k_t) - P_E k_t] \frac{dR_t}{dS_t} + \lambda_t [1 - f(k_t)] \frac{dR_t}{dS_t} - \lambda_t \Big) \qquad [A\text{-}5']$$

where $\dot{\lambda}_t = d\lambda/dt$.

If [A-4′] is used to eliminate one of the unknowns, the result is a system of two differential equations in two unknowns, [A-2] and

[A-5′]. Utilizing the following derivatives, it is possible to consider a phase diagram analysis of this equation system.

$$d\lambda_t/dS_t \Big|_{\dot{\lambda} = 0} = \frac{+\left([P_x f(k_t) - P_E k_t] + \lambda_t[1 - f(k_t)] \right) \frac{d^2R_t}{dS_t^2}}{\rho + 1 - [1 - f(k_t)]\, dR_t/dS_t} \quad \text{[A-6]}$$

$$d\lambda_t/dS_t \Big|_{\dot{S} = 0} = \frac{(P_x - \lambda_t) f'' \left([1 - f(k_t)] \frac{dR_t}{dS_t} - 1 \right)}{R_t f'^2}, \quad \text{[A-7]}$$

$$\frac{d\dot{\lambda}}{dS_t} \Big|_{\lambda\ \text{constant}} = -\left([P_x f(k_t) - P_E k_t] + \lambda_t[1 - f(k_t)] \right) \frac{d^2R_t}{dS_t^2}, \quad \text{[A-8]}$$

$$\frac{d\dot{S}}{d\lambda_t} \Big|_{S_t\ \text{constant}} = -\frac{R_t f'(k_t^2)}{(P_x - \lambda_t) f''(k_t)}. \quad \text{[A-9]}$$

Before proceeding with the analysis, note that:

(i) $f' > 0, f'' < 0, P_x - \lambda_t > 0$ by [A-4′], and $[(P_x - \lambda_t) f(k_t) - P_E k_t + \lambda_t] > 0$ since $P_x - \lambda_t = P_E/f'$ and $f > f' k_t$.

(ii) If a steady state solution exists, such that $\dot{\lambda}_t = 0$ and $\dot{S}_t = 0$, which satisfies the necessary conditions for a maximum, the equilibrium value for S_t, S^*, will be such that $S^* \leqslant \bar{S}_t$ where $\bar{S}_t$ maximizes R_t as a function of S_t. The reason for this was discussed in the section on Notation and Assumptions. It then follows that $dR_t/dS_t > 0$, and $d^2R_t/dS_t^2 < 0$.

(iii) $0 < [1 - f(k_t)] \leqslant 1$ since $S_t = R_t [1 - f(k_t] \geqslant 0$ and $S_t \leqslant R_t$.

(iv) $dR_t/dS_t = 1$ evaluated at S_t^0, where S_t^0 is the spawner stock required for maximum sustained yield; if $S_t < S_t^0$, $dR_t/dS_t > 1$, and if $S_t > S_t^0$, $dR_t/dS_t < 1$.

It is now possible to attach signs to [A-6] through [A-8] as follows:

$$d\lambda_t/dS_t \Big|_{\dot{\lambda} = 0} < 0 \text{ for } S_t^0 \leqslant S_t \leqslant \bar{S}_t,$$

indeterminant for $S_t < S_t^0$; [A-10]

$$d\lambda_t/dS_t \Big|_{\dot{S} = 0} > 0 \text{ for } S_t^0 \leqslant S_t \leqslant \bar{S}_t,$$

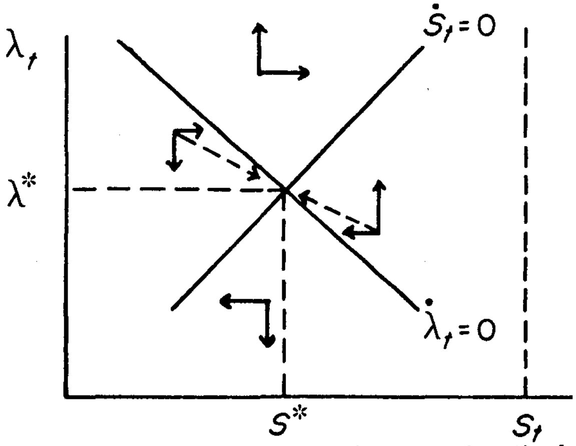
interminant for $S_t < S_t^0$; [A-11]

$$\frac{d\dot{\lambda}}{dS_t} \Big|_{\lambda\ \text{constant}} > 0 \text{ for } S_t \leqslant S_t. \quad \text{[A-12]}$$

$$\frac{d\dot{S}}{d\lambda_t} \Big|_{S_t\ \text{constant}} > 0,\ \forall\ S_t. \quad \text{[A-13]}$$

If $[1 - f(k_t)] \frac{dR_t}{dS_t} \leqslant 1$, then there are no sign ambiguities in the relevant range, and the equilibrium solution exists. From the phase diagram in Appendix Figure 1, it is clear that the equilibrium is a saddle point; i.e., given an initial $S_0 \leqslant \bar{S}_t$, there exists a time path for λ_t, k_t, and S_t converging to the steady state equilibrium, along which the necessary conditions for a maximum are satisfied. Hence, the optimal policy for reaching a steady state solution can be specified even if $S_0 \neq S^*$. Again there is no reason for $S^* = S_t^0$, where S_t^0 corresponds to maximum sustained yield.

It is now possible to calculate the Jacobian determinant J of the system of equations [A-2] and [A-5′] evaluated at the steady state equilibrium, using [A-4′] to eliminate one of the variables. In order to calculate the determinant, the following derivatives are required:



APPENDIX FIGURE 1.—A phase diagram of optimal solution paths.

$$\frac{\partial \dot{\lambda}_t}{\partial \lambda_t} = \rho + 1 - [1 - f(k_t)]\ dR_t/dS_t, \quad \text{[A-14]}$$

$$\partial \dot{S}_t/\partial S_t = \frac{dR_t}{dS_t} 1 - f(k_t)] - 1. \quad \text{[A-15]}$$

Using [A-8], [A-9], [A-14], and [A-15], and assuming that $[1 - f(k_t)]\ \frac{dR_t}{dS_t} \leqslant 1$,

$$J = \begin{vmatrix} \frac{\partial \dot{\lambda}_t}{\partial \lambda_t} & \frac{\partial \dot{\lambda}_t}{\partial S_t} \\ \frac{\partial \dot{S}_t}{\partial \lambda_t} & \frac{\partial \dot{S}_t}{\partial S_t} \end{vmatrix} < 0.$$

If r is substituted for ρ in the determinant, it equivalent to the one discussed in the second section.

As in the second section, it is not possible to prove that the equilibrium solution satisfies sufficient conditions for a maximum, since, in this case, it is not possible either to prove or disprove the concavity of the Hamiltonian.